The Born approximation in the reconstruction in the Calderon inverse problem

J.A. Barceló, **C. Castro**[+], F. Macià and C. Meroño
[+]Univ. Politécnica de Madrid

May 10, 2022

The Calderon problem

Consider the following system in the 3-d unit ball

$$(P1)\begin{cases} -div\,(\gamma(x)\nabla u(x)) = 0, & x \in B, \\ u = f, & x \in \partial\Omega. \end{cases}$$

Given $\gamma \in L^\infty(B)$ we define the Dirichlet to Neumann map (DtN) as

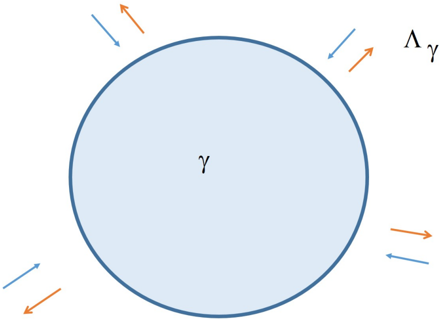
$$\begin{array}{rcl} \Lambda_\gamma : H^{1/2}(\partial B) & \to & H^{-1/2}(\partial B) \\ f & \to & \gamma \frac{\partial u}{\partial n}\Big|_{\partial\Omega} \end{array}$$

The reconstruction problem: Can we find γ from Λ_γ?

Main idea: use a linearization of the map

$$\Lambda_\gamma \to \gamma$$

Mathematical formulation of the classical problem where we want to derive the conductivity from boundary measurements



Multiple applications: EIT, nonintrusive defect detection in materials, geophysics, etc.

Equivalent potential problem

$$(P2)\begin{cases} -\Delta u(x) + q(x)\, u(x) = 0, & x \in B, \\ u = f, & x \in \partial\Omega. \end{cases}$$

Given $q \in L^\infty(B)$ we define the Dirichlet to Neumann map (DtN) as

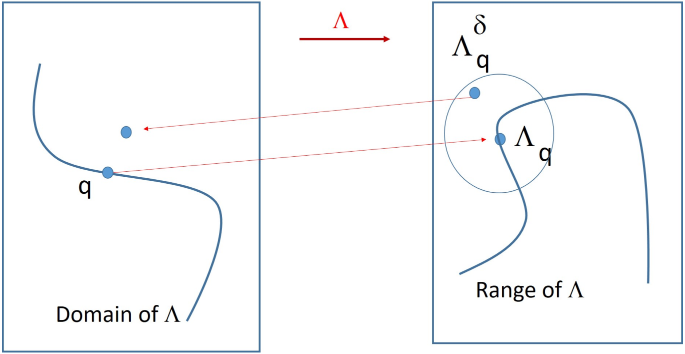
$$\begin{array}{rcl} \Lambda_q : H^{1/2}(\partial B) & \to & H^{-1/2}(\partial B) \\ f & \to & \frac{\partial u}{\partial n} \end{array}$$

The reconstruction problem: Can we find q from Λ_q?
Remark. For smooth γ, (P1) is equivalent to (P2) with

$$q = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}}$$

Ill-posed problem

Close DtN may be associated with very different q, but also one can be outside the domain of Λ [Siltanen and Mueller 2012]



The linearization

In the direct reconstruction numerical algorithm given in [Bikowski, Knudsen, Mueller, 2011], the following Born approximation was also proposed:

$$\widehat{(\gamma_{exp} - 1)}(\xi) = -\frac{2}{|\xi|^2} \lim_{|\zeta|\to\infty} \int_{\partial B_R} e^{-ix\cdot(\xi+\zeta)}(\Lambda_\gamma - \Lambda_1)e^{ix\cdot\zeta}\,dx, \quad \xi \in \mathbb{R}^3,$$

where $\zeta \in \mathcal{V}_\xi = \{\eta \in \mathbb{C}^d \backslash \{0\} \,:\, \eta\cdot\eta = 0,\ (\xi+\eta)^2 = 0\}$.

Remarks:

- It is deduced for smooth γ but can be used for less regular ones
- It is formal in the sense that the limit may not exist, and even if it exists the right hand side may be not the Fourier transform of an L^∞ function
- It requires an high frequency limit which is difficult to compute numerically.
- For $\xi \sim 30$ the right hand side has some inestabilities and we only can compute a (very) low pass filter.

A simplified problem: Radial conductivity $\gamma(x) = \gamma(|x|)$

We assume that γ is a radial function in the unit ball B, i.e. $\gamma(x) = \gamma(|x|)$ and that $\gamma(x) = 1$ in a neighbourhood of the boundary. In this case, $\gamma : [0, 1] \to \mathbb{R}$ is represented by a one-variable function. The following holds:

- The Spherical harmonics are eigenfunctions of Λ_γ. The sequence of eigenvalues $\{\lambda_k[\gamma]\}_{k\geq 0}$ characterize the DtN map Λ_γ
- The eigenvalues of Λ_γ satisfy

$$|\lambda_k[\gamma] - k| \leq Ce^{-k}$$

- It is possible to compute $\lambda_k[\gamma]$ when γ is piecewise constant. For other conductivities $\lambda_k[\gamma]$ can be approximated with a sufficiently close piecewise constant.

Theorem (Barceló, C, Macià and Meroño, 2021 (submitted))

Let $\gamma \in L^\infty(B)$ *a radial function,*

$$\widehat{\gamma^{exp} - 1}(\xi) \;=\; -\pi^{3/2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\Gamma(k+3/2)} \left(\frac{|\xi|}{2}\right)^{2k-2} (\lambda_k[\gamma] - k)\,.$$

Remark

- This formula does not contain a limit and the right hand side is absolutly convergent. However, it is still formal in the sense that we do not know when the right hand side is the Fourier transform of an L^∞ function.
- The series contain terms which are the product of very large and very small numbers. Usual float64 number representation provides an accurate sum only for $\xi \sim 30$, which is the limit found before. Convergence for larger ξ require larger accuracy

Interpretation in terms of moments

It is easy to prove that, when γ is radial,

$$\widehat{\gamma - 1}(\xi) = -\pi^{3/2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\Gamma(k+3/2)} \left(\frac{|\xi|}{2}\right)^{2k-2} \sigma_k[\gamma - 1].$$

where

$$\sigma_k[\gamma] = \int_B (\gamma(x) - 1)|x|^{2k} dx$$

while

$$\widehat{\gamma^{exp} - 1}(\xi) = -\pi^{3/2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\Gamma(k+3/2)} \left(\frac{|\xi|}{2}\right)^{2k-2} (\lambda_k[\gamma] - k).$$

So, γ^{exp} is obtained substituting the moments with $\lambda_k[\gamma] - k$.

Remark. Recovering a functions from its moments is an ill-posed problem. So it is recovering γ^{exp}

Moreover,

$$|\lambda_k[\gamma] - k - \sigma_k[\gamma]| \leq C \frac{\|\gamma\|_{W^{2,\infty}}}{k^3}$$

It is possible to see that $\lambda_k[\gamma] - k$ is relevant for $\xi \sim k/2$. Roughly speaking, the Born approximation approximates better the large frequencies. In scattering theory this is usually translated in a property of recovering singularities in the conductivity.

Radial potentials

Theorem

Let $q \in L^\infty(B)$ a radial function,

$$\widehat{q}(\xi) \quad = \quad 2\pi^{3/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+3/2)} \left(\frac{|\xi|}{2}\right)^{2k} (\lambda_k[\gamma] - k) \,.$$

Idea of the proof:

Given ξ and $h > 0$ let $\zeta_1, \zeta_2 \in \mathbb{C}$ such that

$$\zeta_1 + \zeta_2 = -i\xi h,$$

with $\zeta_1 \cdot \zeta_1 = \zeta_2 \cdot \zeta_2 = 0$, we have that

$$\zeta_1 \cdot \zeta_2 = \frac{1}{2}(\zeta_1 + \zeta_2)^2 = -\frac{1}{2}|\xi|^2 h^2.$$

On the other hand,

$$e_{\zeta/h}(x) = e^{\frac{\zeta}{h}\cdot x} = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{1}{h^k}(\zeta\cdot x)^k,$$

and since in the radial case Λ_q is diagonal in the spherical harmonics, we have that

$$\Lambda_q(e_{\zeta/h})(x) = \sum_{k=0}^{\infty} \frac{\lambda_k}{k!}\frac{1}{h^k}(\zeta\cdot x)^k.$$

Hence,

$$\begin{aligned}\int_{\partial B} e_{\zeta_1/h}(\Lambda_q - \Lambda_0)e_{\zeta_2/h}dx &= \sum_{k=0}^{\infty} \frac{\lambda_k - k}{(h^k k!)^2}\int_{\mathbb{S}^2}(\zeta_1\cdot x)^k(\zeta_2\cdot x)^k\, dS(x)\\ &= \sum_{k=0}^{\infty} c_k\frac{\lambda_k - k}{(h^k k!)^2}\frac{(\zeta_1\cdot\zeta_2)^k}{2^k}\\ &= \sum_{k=0}^{\infty}(-1)^k c_k\frac{\lambda_k - k}{(k!)^2}\left(\frac{|\xi|}{2}\right)^{2k}.\end{aligned}$$

Data simulation: Compute Λ_γ from γ

- Compute a piecewise approximation of $\gamma : [0,1] \to \mathbb{R}$ with 1000 subintervals.
- Approximate the first 400 eigenvalues: $\{\lambda_k(\gamma)\}_{0\leq k\leq 400}$. As $\lambda_k - k \sim e^{-k}$, accurate calculus requires float512 number representation

Born approximation: compute γ_{exp} from $\{\lambda_k(\gamma)\}_{0\leq k\leq 400}$

- Sum the first 400 terms in the series. This give us an approximation of the Fourier tranform in the region

$$\widehat{(\gamma - 1)}(\xi), \quad \xi \in [0, 200]$$

- We invert the Fourier transform in an extended domain $x \in [0, 10]$ to improve precission.
- Here again we need float512 number representation.

Experiment 1: Piecewise constant conductivity

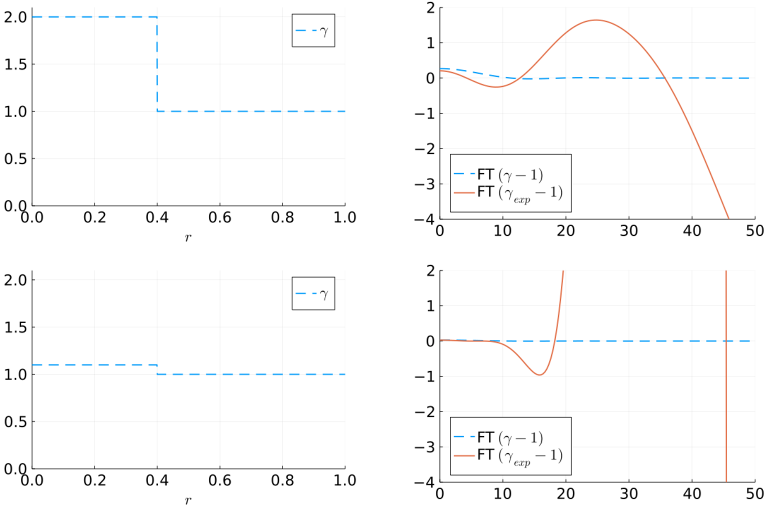
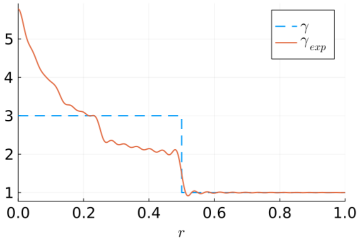
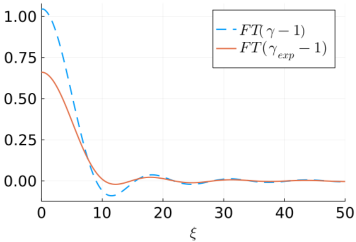


Figure: Experiment 1: A step conductivity (left) and its Fourier transform (right) versus the Fourier transform of its Born approximation (right)

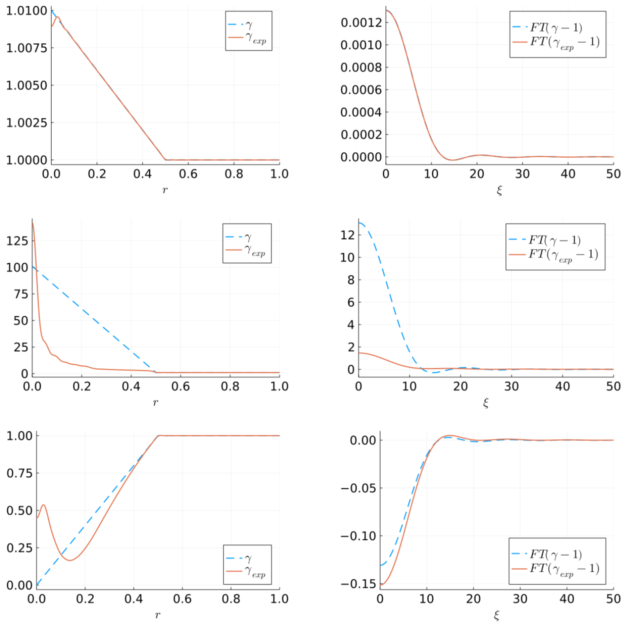
We take

$$\gamma = 2 - \mathrm{sign}(\mathrm{r} - 1/2)\ (r - 1/2)^{0.0001} \sim 2 - \mathrm{sign}(\mathrm{r} - 1/2)(r - 1/2).$$

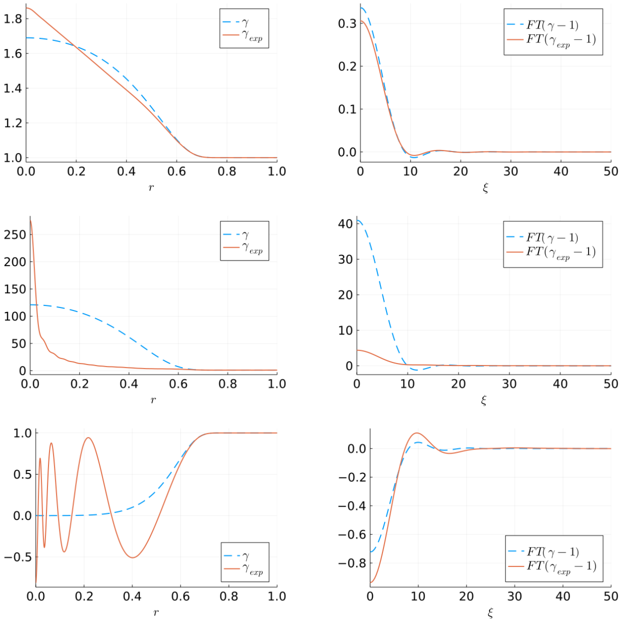




Experiment 3: Lipschitz conductivity



Experiment 4: Smooth conductivity



Experiment 5: Unique continuation from the boundary

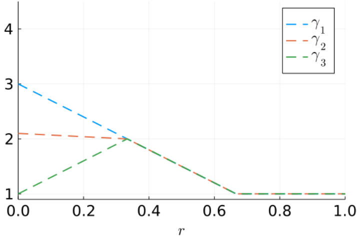


Figure: Experiment 5: Three different conductivities that coincide in the interval $(1/3, 1)$ (left) and their Born approximations (right). We observe that they also coincide in this same interval $(1/3, 1)$.

Experiment 6: Scattering limit

We assume that the domain is the ball of radius $R \to \infty$

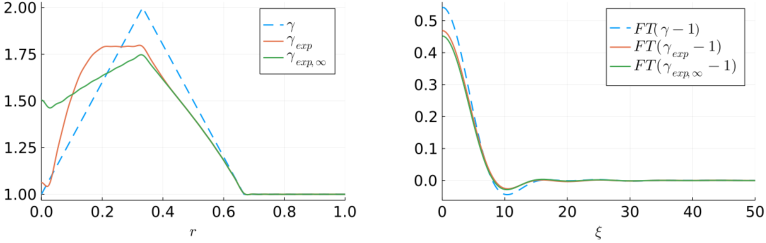


Figure: Experiment 6: Scattering limit.

The Born approximation is more accurate close to the boundary. We have computed the error $e(r) = \frac{1}{N_s}\sum_{i=1}^{N_s} |\gamma(r) - \gamma_{exp}(r)|$ for a random sampling of $N_s = 100$ conductivities.

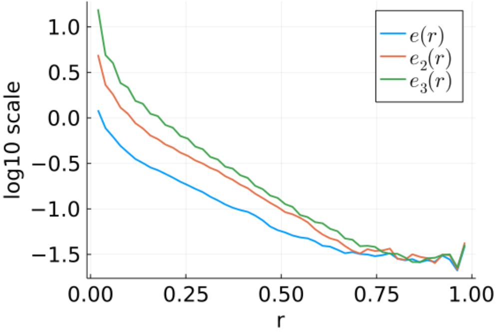


Figure: Experiment 7: average error distribution of the Born approximation, $e_\alpha(x)$, computed with 100 samples, where Fourier coefficients are chosen randomly in different intervals

Fixed point iteration

Based on the good behavior of the Born approximation we propose the following iterative algorithm

$$\begin{cases} \gamma^0 &= \gamma_{exp} \\ \gamma^{n+1} &= \gamma_{exp} + \gamma^n - [\gamma^n]_{\mathrm{exp}} \end{cases}$$

Remarks.

- We initialize with the Born approximation
- We at each step, the DtN map of γ^n must be computed. This requires an approximation by piecewice constant conductivities.

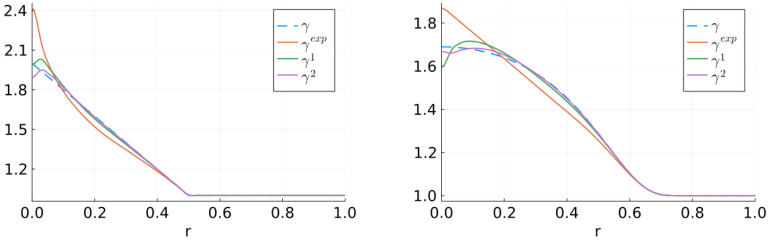


Figure: Example 4: Approximation of the iterative algorithm for a linear and smooth conductivities

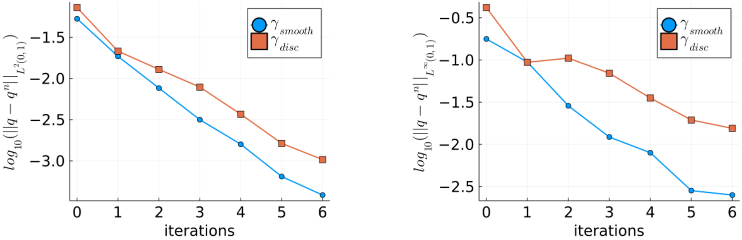


Figure: L^2-error (left) and L^∞-error (right) in log_{10}-scale for the iterative algorithm when considering both the smooth and Lipschitz conductivities

Conclussions

- We have deduced a new formula to compute the Born approximation for radial conductivities. This formula recovers the Fourier transform of γ^{exp} as a series.
- An analogous formula when considering the moments $\sigma_k[\gamma - 1]$, instead of the eigenvalues $\lambda_k[\gamma] - k$ provides the Fourier transform of $\gamma - 1$
- The numerical implementation of the formula requires large precision.
- The Born approximation recovers fairly well the conductivity if γ is not far from 1.
- The Born approximation has some interesting properties as the unique continuation from the boundary
- A suitable fixed point iteration provides a convergent sequence of approximations to the conductivity.

Machine learning method

Main question:

- Design machine learning methods to address the reconstruction problem directly, or based in the above recontruction strategy.

Here we focus on how to adapt Neural ODE to solve the problem

Simplified situation: The radial case

To simplify we assume that q is a radial function, i.e. $q(x) = q(|x|)$. In this case, $q : [0, 1] \to \mathbb{R}$ is represented by a one-variable function that can be approximated in a polynomial, trigonometric or finite element spaces,

$$q(r) \sim \sum_{k \geq 0} q_k \varphi_k(x)$$

Concerning the DtN map we know that the spherical harmonics diagonalize the selfadjoint operator Λ_q and then, this map can be characterized by the eigenvalues

$$\lambda_k[q], \quad k \geq 0$$

The reconstruction in this case consists in approximating the map

$$\{\lambda_k[q]\}_{k \geq 0} \to \{q_k\}_{k \geq 0}$$

First naive approach

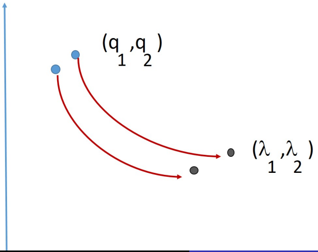
1. Consider an approximation of both q and Λ_q by filtering the higher terms

$$q \sim \{q_k\}_{k \leq N}, \quad \Lambda_q \sim \{\lambda_k[q]\}_{k \leq N}$$

2. Compute a large dataset of sample pairs

$$(\{q_k^s\}_{k \leq N}, \{\lambda_k[q^s]\}_{k \leq N}), \qquad \text{with } s = 1, ..., M$$

3. Train a neural ODE able to construct trajectories between $\{\lambda_k[q]\}_{k \leq N}$ and $\{q_k\}_{k \leq N}$



What is a neural ODE?

It allows to simulate a nonlinear ODE system

$$y' = f(y), \quad y(0) = y_0$$

when f is unknown, based on a large number of solutions.
It is based in a time integration of the ODE with step dt

$$y^{k+1} = y^k + dt\, f(y_k), \quad y^0 = y_0, \quad k = 1, 2, .., K$$

where we change $dt\, f$ by a parametric ansatz

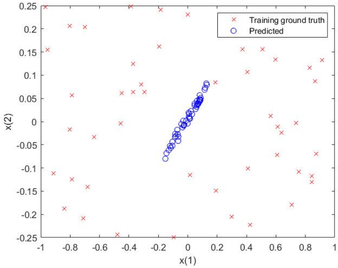
$$y^{k+1} = y^k + A_2\sigma(A_1 y^k + b_1) + b_2$$

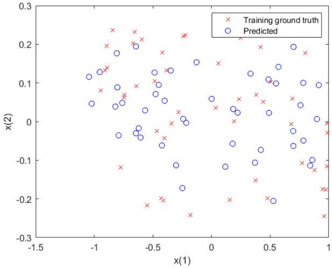
that depends on a nonlinear function σ (tanh here) and some parameters (A_1, A_2 matrixes and b_1, b_2 column vectors), that are optimized according to a loss function, for example

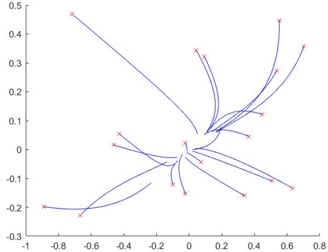
$$loss = \sum_{s=1}^{M} |y^{K,s} - target(s)|.$$

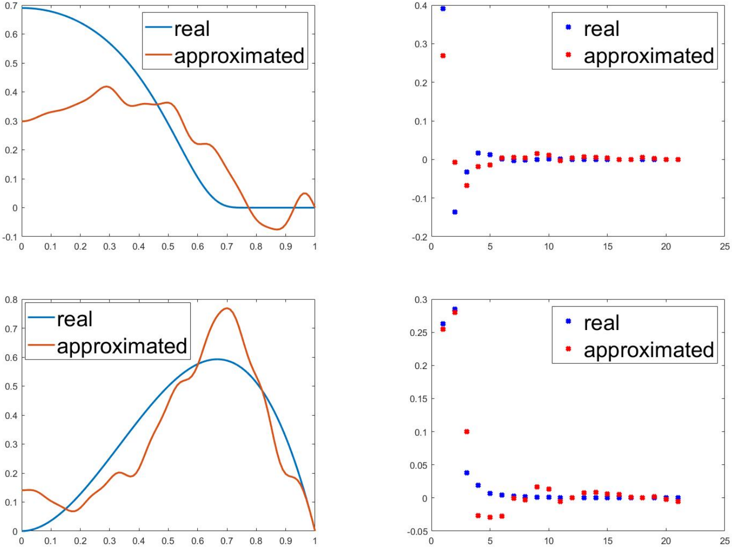
First naive approach: results

We consider $M = 10.000$ samples and $N = 20$ Fourier coefficients. The Neural work has 100 hidden size and 200 time steps.









Dataset: trigonometric basis with random coefficients of the form c_j/j and $c_j \in U[-1,1]$, $j \leq 21$.

$$\begin{array}{ccc} & \mathcal{L} & \\ \{\lambda_k[q]-k\}_{k\geq 0} & \longrightarrow & \widehat{q^{exp}} \\ \{m_k[q]\}_{k\geq 0} & \longrightarrow & \widehat{q} \end{array}$$

Remarks:

- The operator $\mathcal{L}$ provides the Fourier transform of the Born approximation. No limit in ζ is required!
- This allows us to compute a large database of Born approximations very quickly.
- For not too large k, $m_k[q] \sim \lambda_k[q] - k$. This means that approximating the 'first moments' one could improve the Born approximation.
- The operator $\mathcal{L}$ is very unstable. A small perturbation of the sequence produces an unbounded function. It requires high accuracy of the sequence.

Two strategies

$$\{\lambda_k[q]-k\}_{k\geq 0} \xrightarrow{ML} \{m_k[q]\}_{k\geq 0} \xrightarrow{\mathcal{L}} \widehat{q}$$

$$\{\lambda_k[q]-k\}_{k\geq 0} \xrightarrow{\mathcal{L}} \widehat{q^{exp}} \xrightarrow{ML} \widehat{q}$$

Second approach: from eigenvalues to momenta

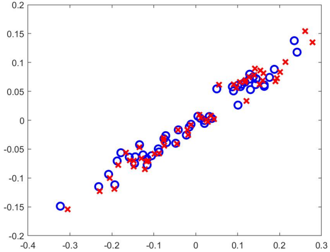
1. Consider an approximation of both the moments of q, $m_k[q]$, and Λ_q by filtering the higher terms

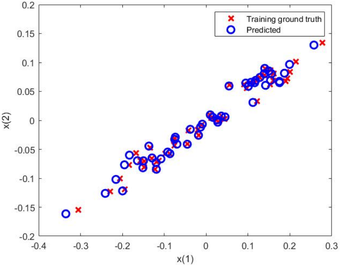
$$q \sim \{m_k[q]\}_{k\leq N}, \quad \Lambda_q \sim \{\lambda_k[q]\}_{k\leq N}$$

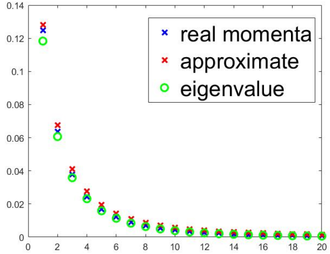
2. Compute a large dataset of sample pairs

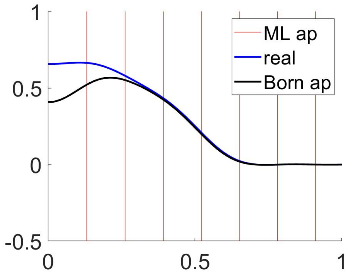
$$(\{m_k[q^s]\}_{k\leq N}, \{\lambda_k[q^s]\}_{k\leq N}), \qquad \text{with } s = 1, ..., M$$

3. Train a neural ODE able to construct trajectories between $\{\lambda_k[q]\}_{k\leq N}$ and $\{m_k[q]\}_{k\leq N}$
4. Compute $\mathcal{L}$









Some regularization is required to compute $\mathcal{L}$!

Third approach: From q^{exp} to q using trigonometric basis

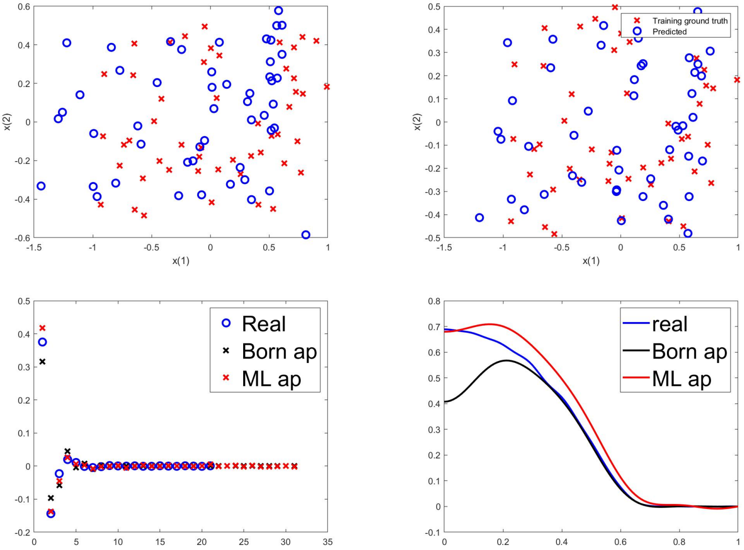
1. Consider an approximation of both q and q^{exp} by filtering the higher terms

$$q \sim \{q_k\}_{k\leq N}, \quad q^{exp} \sim \{q_k^{exp}\}_{k\leq N}$$

2. Compute a large dataset of sample pairs

$$(\{q_k^s\}_{k\leq N}, \{q_k^{exp,s}]\}_{k\leq N}), \quad \text{with } s = 1, ..., M$$

3. Train a neural ODE able to construct trajectories between $\{q_k^s\}_{k\leq N}$ and $\{q_k^{exp,s}]\}_{k\leq N}$



0.6
0.4
0.2
0
-0.2
-0.4
-0.6
x(2)
-1.5
-1
-0.5
0
0.5
1
x(1)
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
Training ground truth
Predicted
Real
Born ap
ML ap
0
5
10
15
20
25
30
35
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
-0.1
-0.2
real

Validation

Distributed error with 1000 validation potentials