Revisit the damped wave equation on torus

Chenmin SUN (CNRS & Université Paris-Est Créteil)

Control & Inverse Problems
Monastir
May 9, 2022

The damped wave equation

Consider the scalar-valued damped wave equation on a compact manifold without boundary:

$$\partial_t^2 u - \Delta_g u + a(x)\partial_t u = 0, \quad (t,x) \in \mathbb{R}_+ \times M,\ u(t,x) \in \mathbb{R},\ a(x) \geq 0.$$

- Energy:

$$E[u](t) := \frac{1}{2}\int_M |\partial_t u|^2 + |\nabla_g u|^2 dx.$$

- Dissipation:

$$\begin{aligned}
\frac{d}{dt}E[u](t) &= \int_M \partial_t u \partial_t^2 u + \nabla_g \partial_t u \cdot \nabla_g u dx \\
&= \int_M \partial_t u(\partial_t^2 u - \Delta_g u) dx \\
&= -\int_M a(x)|\partial_t u|^2 dx \leq 0.
\end{aligned}$$

- Main question: Uniform energy decay rate?

Warming up

The answer is trivial for the constant damping $a(x) \equiv a > 0$:

- Taking the eigenfunction expansion

$$u(t,x) = \sum_{j\geq 0} u_j(t)\varphi_j(x), \text{ where } -\Delta_g \varphi_j = \lambda_j^2 \varphi_j,$$

  we get a completely uncoupled system for $u_j(t)$:

$$u_j''(t) + a u_j'(t) + \lambda_j^2 u_j(t) = 0.$$

Warming up

The answer is trivial for the constant damping $a(x) \equiv a > 0$:

- Taking the eigenfunction expansion

$$u(t,x) = \sum_{j \geq 0} u_j(t)\varphi_j(x), \text{ where } -\Delta_g \varphi_j = \lambda_j^2 \varphi_j,$$

  we get a completely uncoupled system for $u_j(t)$:

$$u_j''(t) + a u_j'(t) + \lambda_j^2 u_j(t) = 0.$$

- In reality, dampings may be effective somewhere and may vanish elsewhere. Determining the relation between the energy decay rate and the (geometric) property of the damping is a stabilization problem in the control theory.

Well-posedness as contraction semigroup

- The damped wave equation $\partial_t^2 u - \Delta_g u + a(x)\partial_t u = 0$ can be written as
$$\partial_t U(t) = \mathcal{A}U(t)$$
on $\mathcal{H} := \dot{H}^1(M) \times L^2(M)$, where $U(t) = (u(t), \partial_t u(t))^t$ and
$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ \Delta_g & -a(x) \end{pmatrix}.$$
- Well-posed by the Hille-Yoshida (Lumer–Phillips) theorem, since $\mathcal{A}$ is maximally dissipative. We denote by $e^{t\mathcal{A}}$, $t \geq 0$ the contraction semigroup.
- By the unique continuation theorem of the Laplace operator, we have $\mathrm{Spec}(\mathcal{A}) \cap i\mathbb{R} = \emptyset$. Indeed, if $i\lambda \in \mathrm{Spec}(\mathcal{A})$ for some $\lambda \in \mathbb{R}$ and $(u, v)^t$ is the corresponding eigenvector (non-constant), then
$$-\Delta u - \lambda^2 u + ia(x)\lambda u = 0.$$
W.L.O.G., assume that $\lambda \neq 0$. Then
$$\int_M a(x)|u|^2 dx = 0, \implies u \equiv 0 \text{ on } \omega := \{a > 0\}.$$
Thus $-\Delta u = \lambda^2 u$, a.e. Thus UCP $\implies u \equiv 0$.

Semigroup decay=Resolvent estimate

- Gearhart's theorem: For the operator $\mathcal{A}$, there is an equivalence between
  (a) $\|e^{t\mathcal{A}}(u_0, v_0)\|_{\mathcal{H}} \leq e^{-ct}\|(u_0, v_0)\|_{\mathcal{H}}$
  (b) $\|R(i\lambda)\|_{\mathcal{L}(\mathcal{H})} \leq C$, for all $\lambda \in \mathbb{R}$.

  Indeed, (b) allows us to continue $R(\lambda)$ analytically to $\mathrm{Re} z > -\delta_0$ for some $\delta_0 > 0$.
- Borichev-Tomilov's theorem: There is an equivalence between
  (a) $\|e^{t\mathcal{A}}(u_0, v_0)\|_{\mathcal{H}} \leq Ct^{-\frac{1}{\alpha}}\|(u_0, v_0)\|_{H^2\times H^1}$, for $t \geq 1$;
  (b) $\|R(i\lambda)\|_{\mathcal{L}(\mathcal{H})} \leq C(1 + |\lambda|)^{\alpha}$, for $\lambda \in \mathbb{R}$.
- However, resolvent estimates and localization property of $\mathrm{Spec}(\mathcal{A})$ are not "equivalent", as $\mathcal{A}$ is non-selfadjoint! There is a large literature concerning the asymptotic behavior of $\mathrm{Spec}(\mathcal{A})$ (Lebeau, Sjöstrand, Hitrik, Nonnenmacher, ...). Here we are only interested in the resolvent estimate (the semigroup decay) for large $|\lambda|$.

How comes the semiclassical analysis

- Originally, the semiclassical analysis was applied to the quantum mechanics to understand the asymptotic behavior as the Plank constant $\hbar \to 0$, by analyzing the semiclassical limit of eigenmodes (quasi eigenmodes) of the semiclassical Schrödinger operator $-\frac{1}{2m}h^2\Delta + V(x)$. It turns out that, in the semiclassical limit, the quantum distribution (semiclassical limit) follows the classical dynamics of the Hamiltonian $H = \frac{|\xi|^2}{2} + V(x)$.

How comes the semiclassical analysis

- Originally, the semiclassical analysis was applied to the quantum mechanics to understand the asymptotic behavior as the Plank constant $\hbar \to 0$, by analyzing the semiclassical limit of eigenmodes (quasi eigenmodes) of the semiclassical Schrödinger operator $-\frac{1}{2m}h^2\Delta + V(x)$. It turns out that, in the semiclassical limit, the quantum distribution (semiclassical limit) follows the classical dynamics of the Hamiltonian $H = \frac{|\xi|^2}{2} + V(x)$.
- It turns out that the resolvent estimate

$$\|R(i\lambda)\|_{\mathcal{L}(\mathcal{H})} \leq C(1+|\lambda|)^{\alpha}$$

is equivalent to the estimate of the form (writing $h = \lambda^{-1}$)

$$\|P_{h,a}^{-1}\|_{\mathcal{L}(L^2)} \leq Ch^{-\alpha-1}, \tag{1}$$

where $P_{h,a} = -h^2\Delta - 1 + iha(x)$.
- This leads to the study of semiclassical limit of quasimodes for the non-selfadjoint perturbation of the free semiclassical Schrödinger operator $P_{h,0} = -h^2\Delta - 1$. For instance, the non-existence of quasi-modes

$$P_{h,a}\psi_h = o_{L^2}(h^{\alpha+1}), \quad \|\psi_h\|_{L^2} = 1$$

implies (1).

Semiclassical limits of ψ_h, $h \to 0$

Semi-classical limits of (ψ_h) are Radon measures on the phase space T^*M.
For the moment, consider a sequence of $o(h)$ quasi-modes:

$$P_{h,a}\psi_h := (-h^2\Delta - 1 + iha(x))\psi_h = o_{L^2}(h).$$

- Existence of semiclassical measures (Gérard, Tartar): Up to a subsequence of $(\psi_h)_{h>0}$, there exists a Radon measure $\mu \geq 0$, such that for any symbol $b \in C_c^\infty(T^*M)$,

$$\langle \mathrm{Op}_h^w(b)\psi_h, \psi_h\rangle_{L^2(M)} \to \int_{T^*M} b(x,\xi)\mu(dxd\xi),$$

  where $\mathrm{Op}_h^w(b)$ is the Weyl-quantization of b, defined locally (on $\mathbb{R}^{2d}$) via

$$\mathrm{Op}_h^w(b)f(x) := \frac{1}{(2\pi h)^d}\int_{\mathbb{R}^{2d}} e^{\frac{i(x-y)\cdot\xi}{h}} b\Big(\frac{x+y}{2}, \xi\Big) f(y)dyd\xi.$$

- The fact that ψ_h are $o(h)$ quasimodes implies:
  - $\mathrm{supp}(\mu) \subset S^*M := T^*M \cap \{|\xi|_g = 1\}$.
  - μ is invariant along the geodesic flow φ_t on S^*M.
  - $a\mu \equiv 0$. This can be deduced from the a priori estimate:

$$\langle a(x)\psi_h, \psi_h\rangle \leq h^{-1}\mathrm{Im}|\langle \psi_h, P_{h,a}\psi_h\rangle|.$$

Semiclassical limits: Sequel

- Any invariant measure μ can be realized as the semiclassical measure of some $O(h)$ quasi-modes of $P_{h,0}$.
- Semiclassical measures of finer quasimodes ψ_h of $P_{h,0}$ (with width $o(h^{1+\alpha})$, $\alpha > 0$) satisfy more constraints. They form a smaller class than the class of invariant measures.
- The resolvent estimate we want to study is equivalent to the following type of uniqueness statement:
  - (S): Assuming that μ is a semiclassical measure for some $o(h^{1+\alpha})$ quasimodes ψ_h of $P_{h,a}$ (which implies that $a\mu \equiv 0$), then $\mu \equiv 0$.
- It turns out that the validity of the above statement depends on the geometry of $\{a(x) > 0\}$ and the regularity of $a(x)$.

Geometric Control Condition (GCC)

Theorem (Rauch-Taylor, Bardos-Lebeau-Rauch, Burq-Gérard)

Assume that $a \in C(M)$ *and* $\omega := \{x : a(x) > 0\}$ *satisfies the* **geometric control condition** *(GCC), then the uniqueness statement (S) is true with* $\alpha = 0$. *Equivalently, we have the resolvent estimate*

$$\|P_{h,0}^{-1}\|_{\mathcal{L}(L^2)} \leq Ch^{-1}.$$

By Gearhart's theorem, the energy of the damped wave equation decays exponentially: $E[u](t) \leq Ce^{-ct}E[u](0)$ *for all* $t \geq 0$.

Geometric Control Condition (GCC)

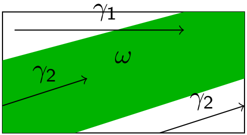
Theorem (Rauch-Taylor, Bardos-Lebeau-Rauch, Burq-Gérard)

Assume that $a \in C(M)$ and $\omega := \{x : a(x) > 0\}$ satisfies the **geometric control condition** *(GCC), then the uniqueness statement (S) is true with $\alpha = 0$. Equivalently, we have the resolvent estimate*

$$\|P_{h,0}^{-1}\|_{\mathcal{L}(L^2)} \leq Ch^{-1}.$$

By Gearhart's theorem, the energy of the damped wave equation decays exponentially: $E[u](t) \leq Ce^{-ct}E[u](0)$ for all $t \geq 0$.

ω satisfies (GCC) means: there exists $T_0 > 0$, such that every geodesics of length $T > T_0$ passes through ω.

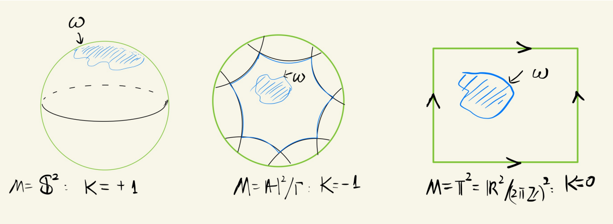


Example of $\mathbb{T}^2$: GCC is satisfied

Beyond GCC: Three model surfaces (compact)

- When $\overline{\omega}$ violates (GCC), the resolvent bound cannot be $O(h^{-1})$, and we need to analyze finer quasi-modes $o(h\epsilon_h)$, $\epsilon_h \to 0$.
- Quasi-modes with smaller width (in order) $o(h\epsilon_h)$ requires finer microlocalization along the geodesics. Then the dynamical property of the geodesic flow φ_t plays an important role!

Beyond GCC: Three model surfaces (compact)



Resolvent bounds for the damped wave operator

ω violates (GCC), $\Vert P_{h,b}^{-1}\Vert_{L^2(M)} \sim g(h)$		
	energy decay rate $f(t)$	resolvent bound $g(h)$
$M=\mathbb{S}^2$	$1/\log(1+t)$	$e^{c/h}$ [1]
$M=\mathbb{H}^2/\Gamma$	$e^{-c't}$ losing derivatives	$h^{-1}\log(h^{-1})$ [2]
$M=\mathbb{T}^2$	$t^{-1/\alpha}$	$h^{-1-\alpha}$, α close to 1 for regular a [3][4][5][6]

References:
[1] Lebeau (1996), [2] Dyatlov-Jin (2018), [3] Burq-Hitrik (2005), [4] Anantharaman-Léautaud (2014), [5] Datchev-Kleinhenz (2020), [6] S. (2021).

Damped wave operator on $M = \mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$

Denote by

$$f(t) = \sup_{(u_0,v_0)\in H^2\times H^1} \frac{\|e^{t\mathcal{A}}(u_0,v_0)\|_{\mathcal{H}}}{\|(u_0,v_0)\|_{H^2\times H^1}}.$$

- When (GCC) is violated by $\overline{\omega}$, where $\omega = \{a > 0\}$ and $\overline{\omega} \neq \mathbb{T}^2$, one cannot expect that the order of $f(t)$ is better than $O(t^{-1})$.
- It turns out that the optimal order of $f(t)$ depends on the vanishing behavior of $a(x)$ near $\Sigma := \partial\{a > 0\}$:
  - Anantharaman-Léautaud 2014: For general $a \in W^{N,\infty}$ and $|\nabla a| \lesssim a^{1-\frac{1}{\beta}}$ with $N, \beta \gg 1$, $f(t) \lesssim t^{-1+\frac{C}{\beta}}$.
    (Earlier contribution by Burq-Hitrik for 1D damping)
  - Datchev-Kleinhenz 2020: When $a(x) = a(x_1) \sim x_1^{\beta}$ near $x_1 = 0$ and $\overline{\omega} \neq \mathbb{T}^2$, $f(t) \sim t^{-1+\frac{1}{\beta+3}}$, optimal!
    (Earlier contribution by R. Stath for the case $\beta = 0$)
- C. S. 2021: It turns out the curvature of $\Sigma = \partial\{a > 0\}$ also plays a role for the optimal decay rate. In particular, the strictly convex damping region can better stabilize the wave!

The Strictly Convex damping region

Theorem (C. S., IMRN, 2022)

Let $\beta > 4$. Assume that the damping $a \geq 0$ has nice vanishing property and the open set $\omega := \{z \in \mathbb{T}^2 : a(z) > 0\}$ is locally strictly convex. Assume that $a(z)$ is locally Hölder of order β near $\partial\omega$, in the sense that there exists $R_0 > 1$, such that

$$\frac{1}{R_0}\mathrm{dist}(z, \partial\omega)^\beta \leq a(z) \leq R_0 \mathrm{dist}(z, \partial\omega)^\beta, \quad \text{for } z \in \omega \text{ near } \partial\omega.$$

Then the energy decay rate of the damped wave equation satisfies $f(t) \leq Ct^{-1+\frac{2}{2\beta+7}}$. Moreover, there are examples of strictly convex dampings $a(x)$ such that the above rate is saturated.

- The decay rate is better than the optimal decay rate $t^{-1+\frac{1}{\beta+3}}$ for the rectangular-shaped ω.

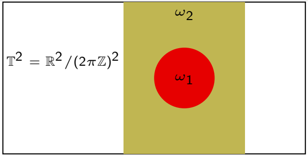
The Strictly Convex damping region

Theorem (C. S., IMRN, 2022)

Let $\beta > 4$. Assume that the damping $a \geq 0$ has nice vanishing property and the open set $\omega := \{z \in \mathbb{T}^2 : a(z) > 0\}$ is locally strictly convex. Assume that $a(z)$ is locally Hölder of order β near $\partial\omega$, in the sense that there exists $R_0 > 1$, such that

$$\frac{1}{R_0}\mathrm{dist}(z, \partial\omega)^{\beta} \leq a(z) \leq R_0\mathrm{dist}(z, \partial\omega)^{\beta}, \quad \text{for } z \in \omega \text{ near } \partial\omega.$$

Then the energy decay rate of the damped wave equation satisfies $f(t) \leq Ct^{-1+\frac{2}{2\beta+7}}$. Moreover, there are examples of strictly convex dampings $a(x)$ such that the above rate is saturated.



The damping a_1 generates better decay rate than a_2

$a_1(x) = (0.1 - |(x_1, x_2)|)_+^{\beta}$, $a_2(x) = (0.5 - |x_1|)_+^{\beta}$

The Strictly Convex damping region

Theorem (C. S., IMRN, 2022)

Let $\beta > 4$. Assume that the damping $a \geq 0$ has nice vanishing property and the open set $\omega := \{z \in \mathbb{T}^2 : a(z) > 0\}$ is locally strictly convex. Assume that $a(z)$ is locally Hölder of order β near $\partial\omega$, in the sense that there exists $R_0 > 1$, such that

$$\frac{1}{R_0}\mathrm{dist}(z, \partial\omega)^{\beta} \leq a(z) \leq R_0 \mathrm{dist}(z, \partial\omega)^{\beta}, \quad \text{for } z \in \omega \text{ near } \partial\omega.$$

Then the energy decay rate of the damped wave equation satisfies $f(t) \leq Ct^{-1+\frac{2}{2\beta+7}}$. Moreover, there are examples of strictly convex dampings $a(x)$ such that the above rate is saturated.

- The same result holds if we replace $\mathbb{T}^2$ to a rectangle $\Omega \subset \mathbb{R}^2$ (with the Dirichlet boundary condition) and ω is a strictly convex open subset of Ω.
- The reason behind is the averaging property of $a(x)$ along the geodesic flow. As the support of a is strictly convex, we gain $1/2$ Hölder regularity locally near the place where $a = 0$.
- Allows to recover the result of Anantharaman-Léautaud with different microlocal techniques.

Reduce to a second microlocalization problem

- Assume that the quasi-modes u_h satisfy

$$(-h^2\Delta - 1 + iha)u_h = o_{L^2}(h^{2+\delta}).$$

- A priori estimates:

$$\int_{\mathbb{T}^2} |h\nabla u_h|^2 - |u_h|^2 = o(h^{2+\delta}), \quad \int_{\mathbb{T}^2} a|u_h|^2 = o(h^{1+\delta}).$$

- Semiclassical measures μ of (u_h) (of some subsequence) is invariant

along the geodesic flow on $\mathbb{T}^2_{x,y} \times \mathbb{S}^2_{\xi,\eta}$. Hence $\mu = 0$ when restricting on irrational directions. To obtain a contradiction, it suffices to show that $\mu|_{\mathbb{T}^2\times\Xi} = 0$ for any rational direction Ξ.

- W.L.O.G, we assume that trapped direction is $(0,1)$ and the quasi-modes are localized of size

$$|D_y| \sim h^{-1}, \quad |D_x| \ll h^{-1}.$$

- We have to do a second microlocalization according to the size of $|D_x|$:
  - TH: $|D_x| \gtrsim h^{-\frac{1+\delta}{2}}$
  - TL: $|D_x| << h^{-\frac{1+\delta}{2}}$

Key points in the proof

• It turns out that the sharp resolvent bound is saturated by transversal low frequencies (TL).

• Using the Birkhoff norm form to average the damping to reduce one dimension in the TL regime.

• Using the known sharp resolvent estimate for the 1D damping (by Datchev-Kleinhenz).

• Key geometric point: averaging along any direction for the convex-shaped damping will gain $1/2$ vanishing order.

• The construction of quasi-modes that saturated the optimal resolvent bound follows from the inverse of the above procedure.

Conclusion and perspectives

- We do not know a simple criterion of the optimal decay rate for the damped wave equation on $\mathbb{T}^2$ in terms of the damping $a(x)$.
- High dimensional torus (or other manifolds): few results.

Thank you for your attention!