Optimal rates of decay for operator semigroups and damped waves

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Control & Inverse Problems

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A class of second-order systems

For Hilbert spaces X_0 and U let

- $A_0 \colon D(A_0) \subseteq X_0 \to X_0$, a positive invertible operator;
- $B_0 \colon U \to X_0$ bounded.

Consider the problem

$$\ddot{u}(t) + B_0 B_0^* \dot{u}(t) + A_0 u(t) = 0, \qquad t \geq 0,$$

to be solved subject to ICs $u(0) \in D(A_0^{1/2})$ and $\dot{u}(0) \in X_0$.

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to be solved subject to ICs $u(0) \in D(A_0^{1/2})$ and $\dot{u}(0) \in X_0$.

Reormulate this as a first-order ACP for $z(t) = (u(t), \dot{u}(t))^T$ on the Hilbert space $X = D(A_0^{1/2}) \times X_0$ with

$$A = \begin{pmatrix} 0 & I \\ -A_0 & -B_0 B_0^* \end{pmatrix}, \qquad D(A) = D(A_0) \times D(A_0^{1/2}).$$

Then A generates a C_0-semigroup $(T(t))_{t \geq 0}$ of contractions on X.

Example 1: The classical damped wave equation

For a smooth domain (or manifold) Ω, let $X_0 = U = L^2(\Omega)$ and

- $A_0 = -\Delta$, the (negative) Dirichlet Laplacian with domain

$$D(A_0) = H^2(\Omega) \cap H_0^1(\Omega);$$

- $B_0 u = b^{1/2} u$ the multiplication operator corresponding to a bounded (continuous) non-negative function b.

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Then the second-order problem becomes

$$u_{tt} + b(x) u_t - \Delta u = 0, \qquad x \in \Omega,\ t > 0,$$

with initial conditions

- $u(\cdot, 0) \in D(A_0^{1/2}) = H_0^1(\Omega)$;
- $u_t(\cdot, 0) \in L^2(\Omega)$.

Energy decay

The **energy** of a solution is given by

$$E(t) = \frac{1}{2}\int_\Omega |\nabla u|^2 + |u_t|^2 \,\mathrm{d}x.$$

The damped wave equation gives

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so the energy decays. Note that

$$E(t) = \frac{1}{2}\left(\|u(\cdot,t)\|^2_{H^1_0(\Omega)} + \|u_t(\cdot,t)\|^2_{L^2(\Omega)}\right) = \frac{1}{2}\|T(t)z_0\|^2_X,$$

where $z_0 = (u(\cdot,0), u_t(\cdot,0))^T \in X = H^1_0(\Omega) \times L^2(\Omega)$.

Example 2: A weakly damped wave equation

Let $X_0 = L^2(0,1)$, $U = \mathbb{C}$ and

- $A_0 = -\partial_x^2$ with domain $D(A_0) = H^2(0,1) \cap H_0^1(0,1)$;
- $B_0 u = bu$ for some $b \in L^2(0,1;\mathbb{R})$.

Then the second-order problem becomes

$$u_{tt} + b(x) \int_0^1 b(r) u_t(r,t)\,\mathrm{d}r - u_{xx} = 0, \qquad x \in (0,1),\ t > 0,$$

with initial conditions

$$z_0 = (u(\cdot,0), u_t(\cdot,0))^T \in X = H_0^1(0,1) \times L^2(0,1).$$

Once again, the energy of a solution is given by

$$E(t) = \frac{1}{2}\|T(t)z_0\|_X^2, \qquad t \geq 0.$$

Stability of C_0-semigroups

Let $(T(t))_{t\geq 0}$ be a contraction semigroup on a Hilbert space X.

We say that $(T(t))_{t\geq 0}$ is

- **strongly stable** if $\|T(t)x\| \to 0$ as $t \to \infty$ for all $x \in X$;
- **semi-uniformly stable** if $\|T(t)(I-A)^{-1}\| \to 0$ as $t \to \infty$;
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$$\text{(U-Stab)} \implies \text{(SU-Stab)} \implies \text{(S-Stab)},$$

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The spectrum of the generator A of $(T(t))_{t\geq 0}$ satisfies

$$\sigma(A) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq 0\}.$$

Of particular importance: the **boundary spectrum** $\sigma(A) \cap i\mathbb{R}$.

Three stability theorems

Let $(T(t))_{t\geq 0}$ be a contraction semigroup on a Hilbert space X, with generator A.

Theorem (Arendt–Batty '88, Lyubich–Vũ '88)

Suppose that $\sigma(A) \cap i\mathbb{R}$ is countable and that $\sigma_p(A) \cap i\mathbb{R}$ is empty. Then $(T(t))_{t\geq 0}$ is strongly stable.

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The semigroup $(T(t))_{t\geq 0}$ is semi-uniformly stable if and only if $\sigma(A) \cap i\mathbb{R}$ is empty.

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Theorem (Gearhart, Prüss, ... 1980s)

The semigroup $(T(t))_{t\geq 0}$ is uniformly stable if and only if $\sigma(A)\cap i\mathbb{R}$ is empty and $\sup_{s\in\mathbb{R}}\|(is-A)^{-1}\|<\infty$.

Returning to our second-order systems

Consider the contraction semigroup $(T(t))_{t\geq 0}$ on the Hilbert space $X = D(A_0^{1/2}) \times X_0$ generated by

$$A = \begin{pmatrix} 0 & I \\ -A_0 & -B_0 B_0^* \end{pmatrix}, \qquad D(A) = D(A_0) \times D(A_0^{1/2}).$$

Assume that A_0 has compact resolvent. Then $\sigma(A) = \sigma_p(A)$, so

$$\text{(S-Stab)} \iff \text{(SU-Stab)} \iff \sigma_p(A) \cap i\mathbb{R} = \emptyset.$$

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$$\text{(S-Stab)} \iff \text{(SU-Stab)} \iff \sigma_p(A) \cap i\mathbb{R} = \emptyset.$$

But $is \in \sigma_p(A) \cap i\mathbb{R}$ if and only if

$$(A_0 - s^2)u + isB_0 B_0^* u = 0$$

has a non-zero solution $u \in D(A_0)$. So $\sigma_p(A) \cap i\mathbb{R} = \emptyset$ if and only if $B_0^* u \neq 0$ for every eigenfunction u of A_0.

Damped waves

Example 1: Classical damping

Here $A_0 = -\Delta$ with Dirichlet BCs on $\partial\Omega$ and $B_0 u = bu$ for $u \in L^2(\Omega)$ and some continuous $b \geq 0$. The condition for semi-uniform stability is that $\|b^{1/2}u\|_{L^2(\Omega)} \neq 0$ all eigenfunctions u of the Dirichlet Laplacian on Ω. This holds whenever $b \neq 0$, by the unique continuation principle.

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Example 2: Weak damping

Here $A_0 = -\partial_x^2$ with Dirichlet BCs at $0, 1$ and $B_0 u = bu$ for $u \in \mathbb{C}$ and some $b \in L^2(0, 1; \mathbb{R})$. The condition for semi-uniform stability becomes

$$\int_0^1 b(x) \sin(n\pi x) \, \mathrm{d}x \neq 0, \qquad n \in \mathbb{N}.$$

Uniform or semi-uniform stability?

Question:

When is $(T(t))_{t\geq 0}$ uniformly stable? That is, when is

$$\sup_{s\in\mathbb{R}} \|(is - A)^{-1}\| < \infty?$$

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For the weakly damped wave equation on $(0,1)$, it is easy to see that uniform stability requires $|b_n| \geq \delta$ for some δ, where

$$b_n := \int_0^1 b(x)\sin(n\pi x)\,\mathrm{d}x, \qquad n\in\mathbb{N}.$$

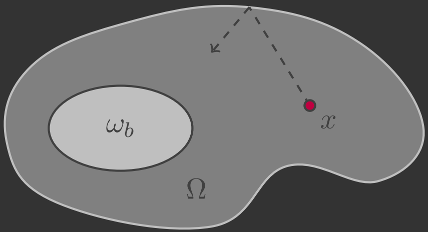
But $(b_n)_{n\geq 1} \in c_0$ as $b\in L^2(0,1)$, so this condition **never** holds.

Uniform stability for classically damped waves

Define the **domain of damping**,

$$\omega_b = \{x \in \Omega : b(x) > 0\}.$$

Given a point $x \in \Omega$, consider billiard ball trajectories starting at x.



We say that ω_b satisfies the **geometric control condition (GCC)** if every billiard ball trajectory eventually intersects ω_b.

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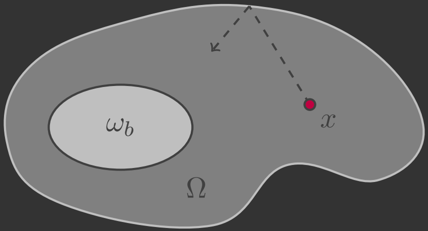

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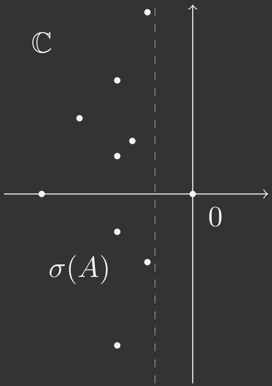
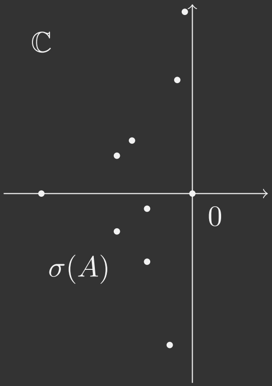
Theorem (Rauch–Taylor '74, Bardos–Lebeau–Rauch '88, Burq–Gérard '97)

We have uniform stability if and only if ω_b satisfies the GCC.

Spectra for the classically damped wave equation





GCC satisfied:

$$\sup_{s\in\mathbb{R}} \|(is - A)^{-1}\| < \infty$$

GCC violated:

$$\sup_{s\in\mathbb{R}} \|(is - A)^{-1}\| = \infty$$

Rates of decay

Let $(T(t))_{t\geq 0}$ be a C_0-semigroup of contractions on a Hilbert space X, with generator A satisfying $\sigma(A) \cap i\mathbb{R} = \emptyset$.

Then $(T(t))_{t\geq 0}$ is semi-uniformly stable, i.e.

$$\|T(t)A^{-1}\| \to 0, \qquad t \to \infty, \tag{1}$$

and $(T(t))_{t\geq 0}$ is uniformly stable if and only if

$$\sup_{s\in\mathbb{R}} \|(is - A)^{-1}\| < \infty. \tag{2}$$

If $(T(t))_{t\geq 0}$ is uniformly stable then it decays exponentially, i.e. there exist $M \geq 0$ and $\omega > 0$ such that

$$\|T(t)\| \leq Me^{-\omega t}, \qquad t \geq 0.$$

What is the rate of decay in (1) when (2) does not hold?

Decay rates for semi-uniform stability

Theorem (Rozendaal–S.–Stahn '19)

Let $(T(t))_{t\geq 0}$ *be a* C_0*-semigroup of contractions on a Hilbert space* X*, whose generator* A *satisfies* $\sigma(A)\cap i\mathbb{R}=\emptyset$*. Suppose that* $\|(is-A)^{-1}\|\leq M(|s|)$ *for all* $s\in\mathbb{R}$*, where* M *has positive increase. Then*

$$\|T(t)A^{-1}\| = O\left(\frac{1}{M^{-1}(t)}\right), \qquad t\to\infty.$$

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$$\|T(t)A^{-1}\| = O\left(\frac{1}{M^{-1}(t)}\right), \qquad t \to \infty.$$

A continuous non-decreasing function $M\colon \mathbb{R}_+ \to (0,\infty)$ has **positive increase** if there exist $c \in (0,1]$, $s_0, \alpha > 0$ such that

$$\frac{M(\lambda s)}{M(s)} \geq c\lambda^\alpha, \qquad \lambda \geq 1,\ s \geq s_0.$$

E.g. $M(s) = Cs^\alpha \log(s)^\beta$ for $C, \alpha > 0$ and $\beta \in \mathbb{R}$.

Optimality of the M^{-1}-theorem

Proposition (Chill–Paunonen–S.–Stahn–Tomilov '21+)

Let $(T(t))_{t\geq 0}$ be a C_0-semigroup of contractions on a Hilbert space X, whose generator A satisfies $\sigma(A)\cap i\mathbb{R}=\emptyset$. Suppose that $M\colon \mathbb{R}_+\to(0,\infty)$ is a continuous non-decreasing function such that $M(s)\to\infty$ as $s\to\infty$ and

$$\limsup_{|s|\to\infty}\frac{\|(is-A)^{-1}\|}{M(|s|)}>0.$$

Then there exists $c>0$ such that

$$\limsup_{t\to\infty} M^{-1}(ct)\|T(t)A^{-1}\|>0, \tag{3}$$

and if M has positive increase then (3) *holds for all $c>0$.*

Optimality of the M^{-1}-theorem (continued)

Proposition (Rozendaal–S.–Stahn '19)

Let $(T(t))_{t\geq 0}$ be a C_0-semigroup of normal contractions on a Hilbert space X whose generator A satisfies $\sigma(A) \cap i\mathbb{R} = \emptyset$. If

$$\|T(t)A^{-1}\| = O\left(\frac{1}{M^{-1}(ct)}\right), \qquad t \to \infty,$$

for some continuous non-decreasing function $M\colon \mathbb{R}_+ \to (0,\infty)$ and some $c > 0$, then M has positive increase.

Classically damped waves: logarithmic decay

For the damped wave equation with $b \neq 0$ we always have

$$\|(is - A)^{-1}\| \leq Ce^{c|s|}, \quad s \in \mathbb{R},$$

for suitable constants $C, c > 0$.

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for suitable constants $C, c > 0$. Hence

$$\|T(t)A^{-1}\| = O\left(\frac{1}{\log(t)}\right), \quad t \to \infty,$$

so for classical solutions

$$E(t) = O\left(\frac{1}{\log(t)^2}\right), \quad t \to \infty.$$

This rate is attained e.g. on the 2-sphere with damping at the poles but no damping in a neighbourhood of the equator.

See [Lebeau '96] and [Burq '98].

Classically damped waves: faster-than-logarithmic decay

- Consider the damped wave equation on $\Omega = (0,1)^2$ (or a torus) and let b be the indicator function of a vertical strip. Then

  $$\|(is - A)^{-1}\| \asymp |s|^{3/2}, \qquad |s| \to \infty,$$

  and hence $\|T(t)A^{-1}\| \asymp t^{-2/3}$ as $t \to \infty$. For classical solutions:

  $$E(t) = O(t^{-4/3}), \qquad t \to \infty.$$

  See [Liu–Rao '07], [Nonnenmacher '14] and [Stahn '17a].

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$$E(t) = O(t^{-4/3}), \qquad t \to \infty.$$
See [Liu–Rao '07], [Nonnenmacher '14] and [Stahn '17a].
- For the wave equation on $\Omega = (0,1)$ with viscoelastic damping at the boundary can get
$$\|(is - A)^{-1}\| \asymp |s|^{\alpha} \log(|s|)^{\beta}, \qquad |s| \to \infty,$$
for any $\alpha \in (0,2)$, $\beta \in \mathbb{R}$. For classical solutions:
$$E(t) = O(t^{-2/\alpha} \log(t)^{2\beta/\alpha}), \qquad t \to \infty.$$
See [Stahn '17b] and [Rozendaal–S.–Stahn '19].

Wave packets for abstract second-order systems

Recall that, for Hilbert spaces X_0, U, a positive invertible operator A_0 on X_0 and a bounded operator $B\colon U \to X_0$, we consider

$$A = \begin{pmatrix} 0 & I \\ -A_0 & -B_0 B_0^* \end{pmatrix}, \quad D(A) = D(A_0) \times D(A_0^{1/2}).$$

Generates a contraction semigroup on $X = D(A_0^{1/2}) \times X_0$.

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Generates a contraction semigroup on $X = D(A_0^{1/2}) \times X_0$.

For $s \geq 0$ and $\delta > 0$, we denote by $\mathrm{WP}_{s,\delta}(A_0^{1/2})$ the spectral subspace of $A_0^{1/2}$ corresponding to the interval $(s-\delta, s+\delta)$. Elements of $\mathrm{WP}_{s,\delta}(A_0^{1/2})$ are called **wave packets**.

If A_0 has compact resolvent then wave packets are linear combinations of eigenvectors corresponding to certain eigenvalues.

From wave packets to resolvent estimates

Theorem (Chill–Paunonen–S.–Stahn–Tomilov '21+)

If $\gamma, \delta \colon \mathbb{R}_+ \to (0,\infty)$ *are bounded functions such that*

$$\|B_0^* u\|_U \geq \gamma(s)\|u\|_{X_0}, \qquad u \in \mathrm{WP}_{s,\delta(s)}(A_0^{1/2}),\ s \geq 0,$$

then $\sigma(A) \cap i\mathbb{R} = \emptyset$ *and there exists a constant* $C > 0$ *such that*

$$\|(is - A)^{-1}\| \leq \frac{C}{\gamma(|s|)^2 \delta(|s|)^2}, \qquad s \in \mathbb{R}.$$

Remark: Related results and conditions have been considered for instance by Ammari, Miller, Tucsnak and Weiss.

Example: The weakly damped wave equation

Consider the weakly damped wave equation

$$u_{tt} + b(x)\int_0^1 b(r)u_t(r,t)\,\mathrm{d}r - u_{xx} = 0, \qquad x \in (0,1),\ t > 0,$$

where $b \in L^2(0,1;\mathbb{R})$, subject to Dirichlet boundary conditions.

If $\delta \in (0,\pi/2)$ then $\mathrm{WP}_{s,\delta}((-\partial_x^2)^{1/2})$ is either empty or spanned by $u_n := \sin(n\pi\cdot)$ for some $n \geq 1$. But

$$B_0^* u_n = \int_0^1 b(x)\sin(n\pi x)\,\mathrm{d}x = b_n, \qquad n \geq 1,$$

so the growth of $\|(is - A)^{-1}\|$ as $|s| \to \infty$ is determined by the rate of decay of the Fourier (sine) coefficients of b.

Example: A particular weakly damped wave equation

Now suppose that $b = \chi_{(0,\xi)}$, where $\xi \in (0,1)$ is an irrational number. To solve:

$$u_{tt} + \chi_{(0,\xi)}(x) \int_0^\xi u_t(r,t)\,\mathrm{d}r - u_{xx} = 0, \qquad x \in (0,1),\ t > 0,$$

subject to Dirichlet boundary conditions. Have $\sigma(A) \cap i\mathbb{R} = \emptyset$.

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subject to Dirichlet boundary conditions. Have $\sigma(A) \cap i\mathbb{R} = \emptyset$.

If ξ is of 'constant type' there exists $c > 0$ such that

$$|b_n| = \left| \int_0^\xi u_n(x)\,\mathrm{d}x \right| \geq \frac{c}{n^3}, \qquad n \geq 1,$$

and hence $\|(is - A)^{-1}\| = O(|s|^6)$ as $|s| \to \infty$. For classical solutions:

$$E(t) = O(t^{-1/3}), \qquad t \to \infty.$$

This estimate is sharp.

Example: Observability of the Schrödinger group

Consider the classically damped wave equation on a domain Ω, with damping coefficient $b \geq 0$.

The corresponding Schrödinger group is **observable** if there exist $C, T > 0$ such that

$$\|u\|_{L^2(\Omega)}^2 \leq C \int_0^T \|b^{1/2} S(t) u\|_{L^2(\Omega)}^2 \, \mathrm{d}t, \qquad u \in L^2(\Omega),$$

where $(S(t))_{t \in \mathbb{R}}$ is the group generated by $-i\Delta$.

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If this holds we may find $c > 0$ such that, for $\delta(s) = c(1+s)^{-1}$,

$$\|b^{1/2} u\|_{L^2(\Omega)} \geq c\|u\|_{L^2(\Omega)}, \qquad u \in \mathrm{WP}_{s,\delta(s)}((-\Delta)^{1/2}),\ s \geq 0.$$

Hence $\|(is - A)^{-1}\| = O(|s|^2)$, so for classical solutions

$$E(t) = O(t^{-1}), \qquad t \to \infty.$$

See also [Anantharaman–Léautaud '14].

Further examples

These include:

- Wave equation on a manifold with 'large' damping away from a submanifold
- Wave equation on $(0,1)$ with damping in one point (requires a more involved version of the theory, where B_0 is ‘unbounded’)
- Damped fractional Klein-Gordon equation on $\mathbb{R}$
- Weakly damped beam equation on $(0,1)$

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- Weakly damped beam equation on $(0, 1)$
- ...

References

Two references:

- J. Rozendaal, D. Seifert, R. Stahn, "Optimal rates of decay for operator semigroups on Hilbert spaces', **Advances in Mathematics**, vol. 346 (2019), pp. 359–388
- R. Chill, L. Paunonen, D. Seifert, R. Stahn, Yu. Tomilov, "Non-uniform stability of damped contraction semigroups", **Analysis & PDE**, accepted for publication (2021)

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Thank you.