Small perturbations of the boundary conditions of an elliptic operator

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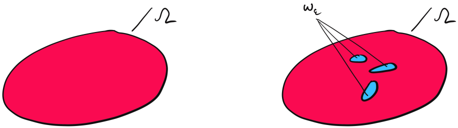
Outline:

1. Introduction/motivation
2. A general representation formula for the first order term
3. Explicit formulas
4. Conclusion

1. Introduction

1.1. Small volume asymptotics

Consider the perturbation of a reference medium with smooth, positive conductivity, by small inhomogeneities



Reference configuration :

conductivity $\gamma(x)$

$$\begin{cases} \operatorname{div}(\gamma \nabla u_0) = 0 & \text{in } \Omega \\ \gamma \nabla u_0 \cdot n = g & \text{on } \partial\Omega \\ \int_{\partial\Omega} u_0 = 0 \end{cases}$$

Perturbed configuration :

conductivity $\gamma_\varepsilon(x) = \gamma(x) + (k - \gamma(x))1_{\omega_\varepsilon}(x)$

$$\begin{cases} \operatorname{div}(\gamma_\varepsilon \nabla u_\varepsilon) = 0 & \text{in } \Omega \\ \gamma_\varepsilon \nabla u_\varepsilon \cdot n = g & \text{on } \partial\Omega \\ \int_{\partial\Omega} u_\varepsilon = 0 \end{cases}$$

There has been a lot of work on deriving asymptotic expansions of $u_\varepsilon - u_0$ or other related quantities

Typically, in the case of a single inclusion, say $\omega_\varepsilon = x_0 + \varepsilon\omega$, one may introduce a Neumann function for the reference PDE

$$\begin{cases} -\mathrm{div}_y(\gamma \nabla_y N(x,y)) & = & \delta_x(y) \quad \text{in } \Omega \\ \gamma \nabla_y N(x,y) \cdot n & = & 1/|\partial\Omega| \quad \text{on } \partial\Omega \end{cases}$$

which, multiplied by the difference $u_\varepsilon - u_0$, yields a representation formula at a point $x \in \Omega$, far from x_0

$$(u_\varepsilon - u_0)(x) \quad = \quad \int_{\omega_\varepsilon} (\gamma - k)\nabla N(x,y) \nabla u_\varepsilon(y)\, dy$$

Introducing the ansatz

$$u_\varepsilon(y) = u_0(y) + \varepsilon v\left(\frac{y - x_0}{\varepsilon}\right) + r_\varepsilon(y)$$

one obtains an expansion of $u_\varepsilon - u_0$ in terms of the volume of ω_ε

$$u_\varepsilon(x) - u_0(x) = |\omega_\varepsilon| M \nabla u_0(x_0) \cdot \nabla N(x, x_0) + o(|\omega_\varepsilon|)$$

The corrector

$$v(y) = \sum_{j=1}^{d} \partial_{x_j} u_0(x_0) \phi_j(y)$$

is a linear combination of the solutions in the whole $\mathbf{R}^d$ to

$$\begin{cases} -\mathrm{div}\,[\gamma_1(y)\nabla(\phi_j(y) + y_j)] = 0 \quad \text{in } \mathbf{R}^n \\ \phi_j(y) \to 0 \quad \text{as } |y| \to \infty \end{cases}$$

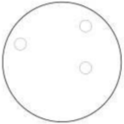
and the *polarization tensor* M is deduced from the correctors via

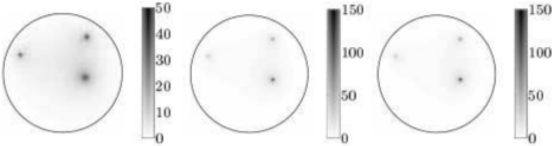
$$M_{i,j} = (\gamma - k) \int_\omega \delta_{ij} - \frac{\partial \phi_i}{\partial y_j}\, dy$$

M contains information on the coefficient contrast and on the geometry of the inhomogeneity

Such expansions have been used successfully to design robust algorithms for the detection of small inhomogeneities from boundary measurements [Brühl Hanke Vogelius, Ammari et al,...]

They also have been used for giving estimates on the volume of the inhomogeneities





References + extensions :

- Conduction [Cedio-Fengya Moskow Vogelius 98]
- Derivation of higher order terms in the expansion for piecewise constant coefficients, using integral equations [Ammari Kang 04]
- Elasticity [Ammari Alves 01]
- Helmholtz equation [Vogelius Volkov 00]
- The Maxwell equations [Ammari Vogelius Volkov 01]
- Asymptotics for eigenvalues [Ammari Moskow]
- Cracks [Friedman Vogelius 89]
- Strip-like inclusions [Beretta Francini Vogelius 03]
- Also work by Kozlov, Movchan, Lipton...

1.2. A general representation formula [Capdeboscq-Vogelius 03]

Assume that the medium is perturbed by inhomogeneities contained in a small subset ω_ε of Ω such that

- ω_ε is measurable
- $\mathrm{dist}(\omega_\varepsilon, \partial\Omega) > d_0 > 0$
- $\lim_{\varepsilon \to 0} |\omega_\varepsilon| \to 0$
- The conductivities γ and γ_ε are uniformly bounded and elliptic

Then for a subsequence $\frac{1}{|\omega_\varepsilon|} 1_{\omega_\varepsilon}(x) \rightharpoonup \mu$ weakly-* in the dual of $\mathcal{C}^0(\overline{\Omega})$

$$\forall\, \phi \in \mathcal{C}^0(\overline{\Omega}), \quad \lim_{\varepsilon \to 0} \int_\Omega \frac{1}{|\omega_\varepsilon|} 1_{\omega_\varepsilon}(x) \phi(x) \quad = \quad < \mu, \phi >$$

there exists a matrix-valued function $\mathcal{M}$ in $L^2(\Omega, d\mu)$ such that

$$u_\varepsilon(x) - u_0(x) \quad = \quad |\omega_\varepsilon| \int_\Omega (\gamma - k) \mathcal{M} \nabla u_0(y) \cdot \nabla N(y, x_0)\, d\mu(y) + o(|\omega_\varepsilon|)$$

Again, the expansion is obtained using the representation formula

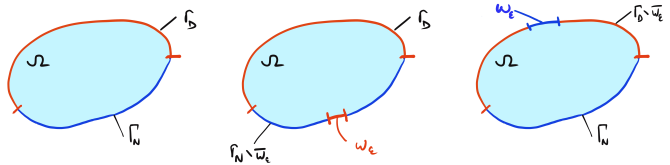
$$\int_\Omega \gamma_0 \nabla(u_\varepsilon - u_0) \cdot \nabla v \quad = \quad \int_{\omega_\varepsilon} (\gamma_0 - \gamma_1) \nabla u_\varepsilon \cdot \nabla v$$

and one can show that

$$\frac{1}{|\omega_\varepsilon|} 1_{\omega_\varepsilon} \nabla u_\varepsilon \, dx \quad \rightharpoonup \quad \mathcal{M} \nabla u_0 \, d\mu$$

- When ω_ε has the form $x_0 + \varepsilon\omega$, then $d\mu = \delta_{x_0}$ and $\mathcal{M} = M\delta_{x_0}$

1.3. Perturbations in the type of BC's



We now consider the following reference configuration :

$$\left\{ \begin{array}{rcll} -\mathrm{div}(\gamma \nabla u_0) & = & f & \text{in } \Omega \\ \frac{\partial u_0}{\partial n} & = & 0 & \text{on } \Gamma_N \\ u_0 & = & 0 & \text{on } \Gamma_D \end{array} \right.$$

and the perturbations

$$\left\{ \begin{array}{rcll} -\mathrm{div}(\gamma \nabla u_\varepsilon) & = & f & \text{in } \Omega \\ \frac{\partial u_\varepsilon}{\partial n} & = & 0 & \text{on } \Gamma_N \setminus \omega_\varepsilon \\ u_\varepsilon & = & 0 & \text{on } \Gamma_D \cup \omega_\varepsilon \end{array} \right. \qquad \left\{ \begin{array}{rcll} -\mathrm{div}(\gamma \nabla u_\varepsilon) & = & f & \text{in } \Omega \\ \frac{\partial u_\varepsilon}{\partial n} & = & 0 & \text{on } \Gamma_N \cup \omega_\varepsilon \\ u_\varepsilon & = & 0 & \text{on } \Gamma_D \setminus \omega_\varepsilon \end{array} \right.$$

where ω_ε is a piece of Γ_N (resp. Γ_D) in the system on the left (resp. on the right)

Previous work

- Perturbations of Neumann BC's relate to the narrow escape problem (motion of a Brownian particle trapped in a domain with reflecting boundary except for narrow absorbing windows)

[Cheviakov-Ward-Straube 10, Holcman-Shuss 14, Ammari-Kalimeris-Kang-Lee 12, Lee-Li-Wang 14,...]

- The case of 2 cavities connected by a narrow gate iz analyzed in [Bendali-Fares-Tizaoui-Tordeux 12]

- There is recent work on optmizing the boundary conditions for wave enhancement in a cavity with tunable reflecting metasurface [Ammari-Imeri-Wu 18, Ammari-Bruno-Imeri-Nigam 20]

where the asymptotic expansions are used to compute a topological derivative in order to nucleate a Neumann part of the boundary

In this latter series of works, perturbed NBC and DBC are addressed for Δ in 2D

Our interest in this problem stems from structural optimization problems in which one tries to distribute a certain amount of material in a fixed given domain Ω so as to minimize a cost functional

A typical example is that of compliance optimization

Given $\partial\Omega = \Gamma_N \cup \Gamma_D, g \in L^2(\Gamma_N)$, $l \in \mathbf{R}$, find $\chi \in L^\infty(\Omega, \{0,1\})$ that minimizes

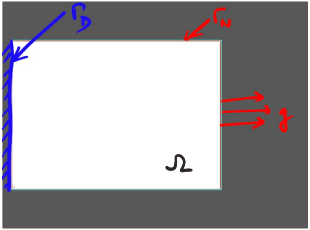
$$J(\chi) \quad = \quad \int_\Omega Ae(u_\chi) : e(u_\chi) + l \int_\Omega \chi$$

where u_χ is the solution in to

$$\begin{cases} \operatorname{div}(Ae(u)) & = & 0 & \text{in } \Omega_\chi := \Omega \cap \{\chi = 1\} \\ Ae(u)n & = & g & \text{on } \Gamma_N \cap \Omega_\chi \\ u & = & 0 & \text{on } \Gamma_D \cap \Omega_\chi \end{cases}$$

where $e(u) := 1/2(\nabla u + \nabla u^T)$ and A denotes the Lamé tensor

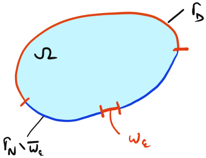
[See the review by G. Allaire, C. Dapogny and F. Jouve, *Shape and Topology Optimization*, Handbook of Numerival Analysis, vol. 22. pp. 1-132, 2021]



How does this relate to the asymptotics ?

- One may want to further rigidify a structure by fixing it to the external frame on small subsets of the boundary (e.g. with screws)
- In additive manufacturing, one tries to avoid overhangs by using thin supporting columns : finding their optimal location may be incorporated in the optimization process
- However, the practical implementation of the asymptotic expansions that we derived requires a more explicit characterization in model situations

Main assumptions for the asymptotics



- Ω is smooth, $\Omega \subset \mathbf{R}^2$ or $\mathbf{R}^3$
- the conductivity γ is elliptic and smooth
- ω_ε is a finite number of connected, open Lipschitz subdomains of $\partial\Omega$, the closures of which do not intersect
- $\omega_\varepsilon \subset \Gamma_N$ and $\mathrm{dist}(\omega_\varepsilon, \Gamma_D) > d_{min} > 0$ for the perturbation of a Neumann BC (and the other way around for the perturbation of a Dirichlet BC)
- f is smooth

2. A general representation formula

2.1. How to measure how small ω_ε is ?

- In the case of the perturbation of a Neumann BC, we set for a set $E \subset \mathbf{R}^d$

$$\begin{aligned}\text{cap}(E) &= \inf\{||v||_{H^1(\mathbf{R}^d)}, \quad v(x) \geq 1 \text{ a.e. in an open neighborhood of } E\} \\ &= \inf\{||v||_{H^1(\mathbf{R}^d)}, \quad v(x) = 1 \text{ a.e. in an open neighborhood of } E\}\end{aligned}$$

Assume that $\Omega \subset \mathbf{R}^d$ is a smooth domain, ω a Lipschitz subset of $\partial\Omega$, and $u \in H^1(\mathbf{R}^d)$ such that

$$u \equiv 1 \quad \text{on } \omega \quad (\text{as a function of } H^{1/2}(\omega))$$

$$\text{then} \qquad \text{cap}(\omega) \leq ||u||^2_{H^1(\mathbf{R}^d)}$$

In particular, if $D_\varepsilon = \{x = (x_1, \ldots, x_{d-1}, 0) \in \mathbf{R}^d, \ |x| < \varepsilon\}$ then

$$\text{cap}(D_\varepsilon) = O(\frac{1}{|ln(\varepsilon)|}) \quad \text{when } d = 2 \qquad \text{cap}(D_\varepsilon) = O(\varepsilon) \quad \text{when } d = 3$$

- In the case of the perturbation of a Dirichlet BC : let $\omega \subset \mathbf{R}^d$ be a finite collection of disjoint Lipschitz hypersurfaces and set

$$e(\omega) = \max_{\kappa} \left\{ \int_{\mathbf{R}^d} |\nabla z|^2 + z^2, \ z \in H^1(\mathbf{R}^d) \setminus \overline{\omega} \text{ and } \left\{ \begin{array}{rcll} -\Delta z + z & = & 0 & \text{in } \mathbf{R}^d \setminus \overline{\omega} \\ \partial_n z & = & \kappa & \text{on } \omega \end{array} \right. \right\}$$

where the max is taken over all functions $\kappa \in \mathcal{C}^\infty(\mathbf{R}^d)$ such that $\kappa(x) = \pm 1, x \in \overline{\omega}$

When ω has a single connected component, $e(\omega)$ is the energy of the unique $z \in H^1(\mathbf{R}^d) \setminus \overline{\omega}$ such that

$$\left\{ \begin{array}{rcll} -\Delta z + z & = & 0 & \text{in } \mathbf{R}^d \setminus \overline{\omega} \\ \partial_n z & = & 1 & \text{on } \omega \end{array} \right.$$

(One could call this number the Neumann capacity of ω)

We would rather use a measure of the smallness of ω_ε that does not require the resolution of a PDE

This is possible in certain cases :

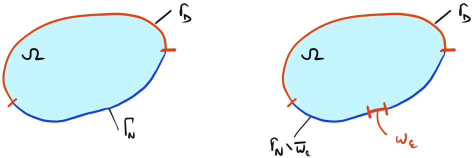
If $\omega \subset \Gamma_D \subset \partial\Omega$ is well separated from Γ_N by d_{min}, then

$$e(\omega) \quad \leq \quad C\,D(\omega) \;:=\; C \int_\Omega \frac{1}{\rho_\omega(x)}\,d\sigma(x)$$

$$\text{where} \quad \rho_\omega(x) \quad := \quad \int_{\partial\Omega\setminus\overline{\omega}} \frac{1}{|x-y|^d}\,d\sigma(y)$$

where $C = C(\Omega, \Gamma_D, d_{min})$

2.2. Perturbation of a Neumann BC



Recall that

$$\left\{\begin{array}{ccll} -\mathrm{div}(\gamma\nabla u_0) & = & f & \text{in } \Omega \\ \frac{\partial u_0}{\partial n} & = & 0 & \text{on } \Gamma_N \\ u_0 & = & 0 & \text{on } \Gamma_D \end{array}\right. \qquad \left\{\begin{array}{ccll} -\mathrm{div}(\gamma\nabla u_\varepsilon) & = & f & \text{in } \Omega \\ \frac{\partial u_\varepsilon}{\partial n} & = & 0 & \text{on } \Gamma_N \setminus \omega_\varepsilon \\ u_\varepsilon & = & 0 & \text{on } \Gamma_D \cup \omega_\varepsilon \end{array}\right.$$

Elliptic regularity theory implies that u_0 is smooth in $\overline{\Omega}$ except possibly at the points where Γ_D, Γ_N and ω_ε meet and

$$||u_0||_{\mathcal{C}^2} \quad \leq \quad C\,||f||_{H^m(\Omega)}$$

Key estimates

Let $\chi_\varepsilon \in H^1(\Omega)$ solve

$$\begin{cases} -\Delta\chi_\varepsilon = 0 & \text{in } \Omega \\ \chi_\varepsilon = 1 & \text{on } \omega_\varepsilon \\ \chi_\varepsilon = 0 & \text{on } \Gamma_D \\ \partial_n \chi_\varepsilon = 0 & \text{on } \Gamma_N \setminus \overline{\omega_\varepsilon} \end{cases}$$

Lemma : There exists constants $0 < c < C$ independent of ω_ε such that

$$c\,\mathrm{cap}(\omega_\varepsilon)^{1/2} \leq ||\chi_\varepsilon||_{H^1(\Omega)} \leq C\,\mathrm{cap}(\omega_\varepsilon)^{1/2}$$

$$||\chi_\varepsilon||_{L^2(\Omega)} \leq C\,\mathrm{cap}(\omega_\varepsilon)^{3/4}$$

It follows from these estimates, that for any function $\phi \in \mathcal{C}^1(\partial\Omega)$ (continuously extended into Ω by a function $\tilde{\phi}$)

$$|\int_{\partial\Omega} \chi_\varepsilon \partial_n \chi_\varepsilon \phi \, d\sigma(y)| \quad = \quad |\int_{\Omega} \nabla(\chi_\varepsilon \tilde{\phi}) \cdot \nabla \chi_\varepsilon \, dy| \leq C \, \text{cap}(\omega_\varepsilon) \, ||\phi||_{\mathcal{C}^1(\partial\Omega)}$$

Invoking the Banach-Alaoglu Theorem, we may assume that there exists a bounded linear functional μ on $\mathcal{C}^1(\partial\Omega)$ such that (for a subsequence)

$$\int_{\partial\Omega} \frac{1}{\text{cap}(\omega_\varepsilon)} \chi_\varepsilon \partial_n \chi_\varepsilon \phi \, d\sigma(y) \quad \to \quad < \mu, \phi >$$

Consider the difference $r_\varepsilon = u_\varepsilon - u_0$, solution to

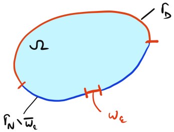
$$\begin{cases} -\mathrm{div}(\gamma \nabla r_\varepsilon) = 0 & \text{in } \Omega \\ r_\varepsilon = -u_0 & \text{on } \omega_\varepsilon \\ r_\varepsilon = 0 & \text{on } \Gamma_D \\ \partial_n r_\varepsilon = 0 & \text{on } \Gamma_N \setminus \omega_\varepsilon \end{cases}$$

Since u_0 is smooth on ω_ε (as these sets are uniformly embedded in Γ_N) the Lemma shows that

$$||r_\varepsilon||_{H^1(\Omega)} = O(\mathrm{cap}(\omega_\varepsilon)^{1/2}) \quad \text{and} \quad ||r_\varepsilon||_{L^2(\Omega)} = O(\mathrm{cap}(\omega_\varepsilon)^{3/4})$$

Using the Neumann function N, we obtain the representation formula

$$
\begin{aligned}
r_\varepsilon(x) &= \int_\Omega \big(-\operatorname{div}(\gamma(y)\nabla_y N(x,y))\big) r_\varepsilon(y)\, dy \\
&= \int_\Omega \gamma(y)\nabla r_\varepsilon(y)\cdot\nabla_y N(x,y)\, dy \;-\; \int_{\partial\Omega} \gamma(y)\partial_{n_y} N(x,y)\, r_\varepsilon(y)\, d\sigma(y) \\
&= \int_{\partial\Omega} \gamma(y)\partial_n r_\varepsilon(y)\, N(x,y)\, d\sigma(y) \;-\; \int_{\partial\Omega} \gamma(y)\partial_{n_y} N(x,y)\, r_\varepsilon(y)\, d\sigma(y) \\
r_\varepsilon(x) &= \int_{\partial\Omega} \gamma(y)\partial_n r_\varepsilon(y)\, N(x,y)\, d\sigma(y)
\end{aligned}
$$



Let $\phi \in \mathcal{C}^1(\partial\Omega)$ such that $\phi = 0$ on $\{y \in \partial\Omega, \operatorname{dist}(y,\Gamma_D) < d_{min}/3\}$

We extend ϕ inside Ω by a function ψ so that

$$\psi = \phi \text{ on } \partial\Omega, \quad \text{and} \quad ||\psi||_{\mathcal{C}^1(\overline{\Omega})} \leq C\,||\phi||_{\mathcal{C}^1(\partial\Omega)}$$

Then we compute (compensated compactness argument [Murat-Tartar, Capdeboscq-Vogelius])

$$\begin{aligned}
\int_{\partial\Omega} \gamma \partial_n r_\varepsilon\, \phi &= \int_{\omega_\varepsilon} \gamma \partial_n r_\varepsilon\, (\chi_\varepsilon \psi) \\
&= \int_{\Omega} \gamma \nabla r_\varepsilon \cdot \nabla(\chi_\varepsilon \psi) \\
&= \int_{\Omega} \gamma \chi_\varepsilon \nabla r_\varepsilon \cdot \nabla \psi \; + \; \int_{\Omega} \gamma \psi \nabla r_\varepsilon \cdot \nabla \chi_\varepsilon \\
&= \int_{\Omega} \nabla(\gamma \psi r_\varepsilon) \cdot \nabla \chi_\varepsilon + T_1 - \int_{\Omega} r_\varepsilon \nabla(\gamma\psi) \cdot \nabla \chi_\varepsilon \\
&= \int_{\partial\Omega} (\partial_n \chi_\varepsilon) \left[\gamma \psi r_\varepsilon\right] + T_1 + T_2 \qquad (\text{and } r_\varepsilon = -u_0 \text{ on } \omega_\varepsilon) \\
&= \int_{\omega_\varepsilon} (\partial_n \chi_\varepsilon) \left[\gamma(-u_0 \chi_\varepsilon)\phi\right] + T_1 + T_2 \\
&= -\int_{\partial\Omega} \chi_\varepsilon \partial_n \chi_\varepsilon \left[\gamma u_0 \phi\right] + T_1 + T_2
\end{aligned}$$

The terms T_1 and T_2 can be estimated using the bounds on χ_ε and the regularity of u_0

$$\begin{aligned} |T_1| &\leq C\,||\chi_\varepsilon||_{L^2(\Omega)}||r_\varepsilon||_{H^1(\Omega)}||\psi||_{\mathcal{C}^1(\overline{\Omega})} \\ &\leq C\,\mathrm{cap}(\omega_\varepsilon)^{5/4}||f||_{H^m(\Omega)}||\phi||_{\mathcal{C}^1(\partial\Omega)} \end{aligned}$$

and a similar estimate holds for T_2.

Let $x \in \Omega$ and η a smooth cut-off function on $\partial\Omega$ with support in Γ_N

We set $\phi(y) = N(x,y)\eta(y)$ to rewrite the representation formula

$$\begin{aligned} r_\varepsilon(x) &= \int_{\partial\Omega} \gamma(y)\partial_n r_\varepsilon(y)\, N(x,y)\, d\sigma(y) \\ &= \int_{\partial\Omega} \gamma(y)\partial_n r_\varepsilon(y)\, N(x,y)\eta(y)\, d\sigma(y) \\ &= -\int_{\partial\Omega} \chi_\varepsilon \partial_n \chi_\varepsilon \left[\gamma u_0 N(x,y)\eta\right] d\sigma(y) \;+\; O(\mathrm{cap}(\omega_\varepsilon)^{5/4}) \\ &= -\mathrm{cap}(\omega_\varepsilon)\; < \mu, \gamma u_0 N(x,\cdot)\eta > + o(\mathrm{cap}(\omega_\varepsilon)) \end{aligned}$$

One can check that

$$< \mu, 1 > \quad = \quad \lim_{\varepsilon \to 0} \int_{\partial\Omega} \chi_\varepsilon \, \partial_n \chi_\varepsilon, d\sigma(y) \; > \; 0$$

and that the support of μ is included in any compact subset of $\partial\Omega$ that contains all the ω_ε's, so that μ is in fact a non-trivial Radon measure

Thm

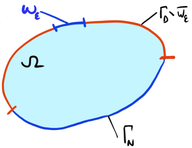
Let u_ε solve the Perturbed-NBC problem and assume that $\lim_{\varepsilon\to 0} \mathrm{cap}(\omega_\varepsilon) = 0$

There exists a subsequence of the u_ε's and a Radon measure μ supported in $\partial\Omega$, such that for all $x \in \Omega$

$$u_\varepsilon(x) \quad = \quad u_0(x) - \mathrm{cap}(\omega_\varepsilon) < \mu, \gamma u_0 N(x, \cdot) > \; + \, o(\mathrm{cap}(\omega_\varepsilon))$$

The first order term represents the field induced by a collection of monopoles distributed on the limiting location of the vanishing subsets ω_ε

2.3. Perturbation of a Dirichlet BC



We can analyse this case using the same ideas : what replaces the functions χ_ε here are the solutions in $H^1(\Omega)$ to

$$\begin{cases} -\Delta\zeta_\varepsilon = 0 & \text{in } \Omega \\ \zeta_\varepsilon = 0 & \text{on } \Gamma_D \setminus \omega_\varepsilon \\ \partial_n \zeta_\varepsilon = 1 & \text{on } \omega_\varepsilon \\ \partial_n \zeta_\varepsilon = 0 & \text{on } \Gamma_N \end{cases}$$

Lemma : There exists constants $0 < c < C$ independent of ω_ε such that

$$c\, e(\omega_\varepsilon)^{1/2} \leq ||\zeta_\varepsilon||_{H^1(\Omega)} \leq C\, e(\omega_\varepsilon)^{1/2}$$

$$||\zeta_\varepsilon||_{L^2(\Omega)} \leq C\, e(\omega_\varepsilon)^{3/4}$$

Thm

Let u_ε solve the Perturbed-DBC problem and assume that $\lim_{\varepsilon \to 0} e(\omega_\varepsilon) = 0$

There exists a subsequence of the u_ε's and a Radon measure μ supported in $\partial\Omega$ such that for all $x \in \Omega$

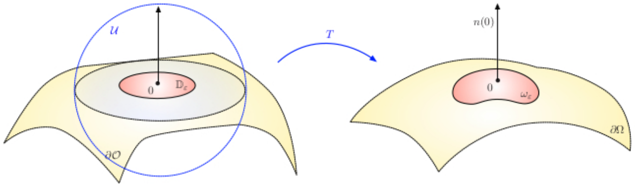
$$u_\varepsilon(x) \quad = \quad u_0(x) - e(\omega_\varepsilon) < \mu, \gamma(y) \frac{\partial u_0(y)}{\partial n} \frac{\partial N(x,y)}{\partial_n} > \; + \, o(e(\omega_\varepsilon))$$

In this case, the first order term represents the field induced by a collection of dipoles distributed on the limiting location of the vanishing subsets ω_ε

4. Explicit examples

It is easy to check that the measures μ are Radon measures, the supports of which are contained in any compact subset of $\partial\Omega$ that contains the ω_ε's

We have investigated simple situations, when the ω_ε's are surfacic balls, where we have been able to give an explicit characterization of the corresponding μ's



Specifically, ω_ε is the image of a segment (in 2d) or of a flat disc (in 3d) that shrinks to a point (say 0) as $\varepsilon \to 0$

Thm

Assume that the sets ω_ε are unifomly contained in the interior of either Γ_N or Γ_D and concentrate on the point 0

Assume that $x \in \overline{\Omega} \setminus \left[(\Gamma_D \cap \Gamma_N) \cup \{0\}\right]$

Expansion for the perturbed NBC

$$u_\varepsilon(x) = u_0(x) - \frac{\pi}{|ln(\varepsilon)|}\gamma(0)u_0(0)N(x,0) + o(\frac{1}{|ln(\varepsilon)|}) \quad \text{when } d = 2$$

$$u_\varepsilon(x) = u_0(x) - 4\varepsilon\gamma(0)u(0)N(x,0) + o(\varepsilon) \quad \text{when } d = 3$$

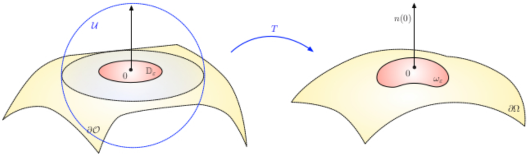
Expansion for the perturbed DBC

$$u_\varepsilon(x) = u_0(x) + a_d\varepsilon^d\gamma(0)\frac{\partial u_0(0)}{\partial n}\frac{\partial N(x,y)}{\partial n_y} + o(\varepsilon^d)$$

with $d = 2, 3$, $a_2 = \pi/2$ and $a_3 = 1/3$

A few words about the proof

Pull back ω_ε to a locally flat interface D_ε with $T\ : O \to \Omega$



Set $\quad v_\varepsilon = u_\varepsilon \circ T, \quad v_0 = u_0 \circ T$

The PDE $\quad -\mathrm{div}(\gamma \nabla u_\varepsilon) = f \quad$ transforms into $\quad -\mathrm{div}(A \nabla v_\varepsilon) = f \circ T$

with a non-homogeneous, possibly anisotropic conductivity

$$A = |\det(\nabla T)|(\gamma \circ T) \nabla T^{-1} \nabla T^{-T}$$

Given a symmetric matrix $\tilde{A}$ set $\tilde{M} = \tilde{A}^{-1/2}$ and

$$L_{\tilde{A}}(x,y) \quad = \quad -\frac{1}{2\pi}|\det(\tilde{M})|\big(\ln|\tilde{M}x - \tilde{M}y| + \ln|\tilde{M}x - \tilde{M}y + 2y_2|\big)$$

Then, in the lower half-space $H = \{y_2 < 0\}$, $L_{\tilde{A}}$ solves

$$\begin{cases} -\mathrm{div}(\tilde{A}\nabla L_{\tilde{A}}(x,y) & = & \delta_x(y) & \text{in } H \\ \tilde{A}\nabla L_{\tilde{A}}(x,y)\cdot n(y) & = & 0 & \text{on } \partial H \end{cases}$$

For $x \in O$, away from D_ε we choose $\tilde{A} = A(x)$ and we write a representation formula for $s_\varepsilon = v_\varepsilon - v_0$

$$
\begin{aligned}
s_\varepsilon(x) &= -\int_O \operatorname{div}(A(x)(\nabla_y L_{A(x)}(x,y))s_\varepsilon(y) \\
&= -\int_{\partial O\setminus U} A(x)\nabla_y L_{A(x)}\cdot n s_\varepsilon + \int_O (A(x)-A(y))\nabla_y L_{A(x)}\cdot\nabla_y s_\varepsilon \\
&\quad + \int_{\Gamma_D} (A\nabla s_\varepsilon\cdot n)(y)\, L_{A(x)}\, d\sigma(y) + \int_{D_\varepsilon} (A\nabla s_\varepsilon\cdot n)(y)\, L_{A(x)}\, d\sigma(y)
\end{aligned}
$$

Letting $x \to D_\varepsilon$ and rescaling, we obtain an integral equation for

$$\varphi_\varepsilon(z) = (A\nabla s_\varepsilon\cdot n)(\varepsilon z), \qquad z \in D_1$$

$$T_\varepsilon\varphi_\varepsilon := \int_{D_1} \varphi_\varepsilon(z) L_A(\varepsilon x)(\varepsilon x, \varepsilon z)\, d\sigma(z) = -u_0(0) + \eta_\varepsilon$$

where the kernel is explicit

We show that $\eta_\varepsilon \to 0$ in $H^{1/2}(D_\varepsilon)$

and that

$$T_\varepsilon \varphi \quad \sim \quad \frac{1}{\pi\gamma(0)}\left(|\ln(\varepsilon)| + cste\right) \int_{D_1} \varphi \, d\sigma(y) \; + \; \frac{2}{\gamma(0)} S_1 \varphi$$

where the operator S_1 is invertible on $H^{-1/2} \to H^{1/2}$

$$S_1\varphi(x) \quad = \quad \frac{1}{\pi} \int_{D_1} \ln|x - z| \varphi(z) \, d\sigma(z)$$

This requires estimating the integral operator with kernel

$$K_\varepsilon(x,z) = \frac{1}{\pi\gamma(0)}\ln|z| - \frac{1}{\pi\sqrt{|\det(A(\varepsilon x)|}}\ln\left|\sqrt{\gamma(0)}M(\varepsilon x)z\right|$$

which is a homogeneous of class (-1)

The associated integral operator maps $H^{-1/2}(D_1)$ into $H^{1/2}(D_1)$

Moreover

$$\lim_{\varepsilon\to 0}\sup_{|\alpha|,|\beta|\le k}\sup_{x\in\mathbf{R}^2}\sup_{|z|=1}\left|\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^\beta}{\partial z^\beta}K_\varepsilon(x,z)\right| = 0$$

which shows that this operator tends to 0 in the operator norm

We rewrite the integral equation as

$$
\begin{aligned}
\underset{\substack{\downarrow \\ 0}}{-u_0(0) + \eta_\varepsilon} &= T_\varepsilon \varphi_\varepsilon(x) \\
&= \int_{D_1} \varphi_\varepsilon(z) L_A(\varepsilon x)(\varepsilon x, \varepsilon z)\, d\sigma(z) \\
&= \int_{D_1} \varphi_\varepsilon(z) \frac{1}{\pi\gamma(0)} \ln|\varepsilon x - \varepsilon z|\, d\sigma(z) \; + \; R_\varepsilon \varphi_\varepsilon(x) \\
&= \frac{1}{\pi\gamma(0)} (|\ln\varepsilon| + cste) \int_{D_1} \varphi_\varepsilon + \underbrace{\frac{1}{\pi\gamma(0)} \int_{D_1} \varphi_\varepsilon \ln|x - z|\, d\sigma(z)}_{\text{invertible}} + \underbrace{R_\varepsilon \varphi_\varepsilon}_{0}
\end{aligned}
$$

So we get

$$
\int_{D_1} \varphi_\varepsilon = \frac{-\pi\gamma(0) u_0(0)}{|\ln\varepsilon|} + o\left(\frac{1}{|\ln\varepsilon|}\right)
$$

Finally, we write another representation formula for u_ε, using the fundamental solution of the reference configuration

$$u_\varepsilon(x) = u_0(x) + \int_{\omega_\varepsilon} \gamma(y) \frac{\partial}{\partial_n}(u_\varepsilon - u_0)(y) N(x,y)\, d\sigma_y$$

which after pull-back and rescaling yields

$$\begin{aligned} u_\varepsilon(x) &= u_0(x) + \int_{D_1} \varphi_\varepsilon(z) N(x, T\varepsilon z)\, d\sigma_z \\ &= u_0(x) + \left(\int_{D_1} \varphi_\varepsilon(z)\, d\sigma_z\right) N(x,0) \ + \ O(\varepsilon) \\ &= u_0(x) - \frac{\pi}{|ln(\varepsilon)|} \gamma(0) u_0(0) N(x,0) \ + \ o(\frac{1}{|ln(\varepsilon)|}) \end{aligned}$$

- [Cheviakov, Ward, Li,...] used matched asymptotics for the perturbed NBC
- A similar integral representation was used in the work of [Ammari et al]
- See also [Bonnet] for the case of the Lamé operator

Conclusion

- We derived a general representation formula for the asymptotics of the solution to an elliptic PDE when the type of BC is changed over a small subset ω_ε of the boundary

- We obtained an explicit characterization of the first order term when ω_ε is a surfacic ball

- These asymptotics generalize those derived for small internal inhomogeneities, and the general representation formula shows a similar structure

- Perspectives : higher order terms for the perturbed Dirichlet BC, [Ammari-Kalameris-Kang-Lee], study of effective boundary behaviors

- Application to shape optimization: we want to use these asymptotic to compute a topological derivative in order to optimize also the places where a structure would be attached