Partial differential difference equations PDDEs

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Abstract delay system

Let $(Y, \|.\|_y)$ be a Banach space and $X = C([-\tau, 0], Y)$ the banach space endowed with the uniform norm

$$\|f\|_X = \sup_{t\in[-\tau,0]} \|f(t)\|_Y$$

Consider the following dynamical system

$$\begin{cases} \dot{x}(t) & = & Bx(t) + \phi x_t, t \geq 0 \\ x_0 & = & \varphi \in X \end{cases} \tag{1}$$

where

- $x_t : [-\tau, 0] \to Y$ defined by $x_t(s) = x(t+s)$.
- $\phi \in \mathcal{L}(X, Y)$ is the delay bounded operator.
- $(B, \mathcal{D}(B))$ is the generator of a C^0 semigroup on Y.
- φ is the initial data.

Definition

A continuous function $x : [-\tau, +\infty[\to Y$ is called a solution of (1) if all the following properties hold:

- x is right-sided differentiable at $t = 0$ and continuously differentiable for all $t > 0$.
- $x(t) \in \mathcal{D}(B)$ for all $t \geq 0$.
- x satisfies (1).

Let $(A, \mathcal{D}(A))$ be the corresponding delay differential operator on X defined by

$$\begin{cases} A(f) = \dot{f}, \\ \mathcal{D}(A) = \{f \in C^1([-\tau, 0], Y); f(0) \in \mathcal{D}(B); \dot{f}(0) = Bf(0) + \phi f\}. \end{cases} \tag{2}$$

Well posedness

Theorem

- The operator $(A, \mathcal{D}(A))$ in (2) generates a C^0 semigroup $(T(t))_{t\geq 0}$ on X.
- If $\varphi \in \mathcal{D}(A)$, then the function $x : [-\tau, 0, +\infty[\to Y$ defined by

$$\begin{cases} x(t) = \varphi(t) \; if \; -\tau \leq t \leq 0 \\ x(t) = [T(t)\varphi](0) \; if \; t > 0 \end{cases} \tag{3}$$

is the unique solution of (1).

Stability of system (1) based on Lyapunov-Razumikhin theory

Theorem

Suppose $u, v, : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are continuous, nondecreasing functions, $u(s), v(s)$ positive for $s > 0$ and $u(0) = v(0) = 0$. If there are $c > 0$ and a continuous function $V : Y \rightarrow \mathbb{R}$ such that

$$u(\|x\|) \leq V(x,t) \leq v(\|x\|)$$

and

$$\dot{V}(x,t) \leq -cV(x,t), \; \textit{if} \; V(x(s,t+\theta)) \leq V(x(s,t)), \; \textit{forall} \, \theta \in [-\tau, 0],$$

then system (1) is exponentially stable with decay rate equal to c.

Motivation

- delay is present in many dynamical systems such as Population dynamics, nonlinear optics and fluid dynamics.
- Considering delay in control systems comes from the observation that for many control systems, the dalay is not only inevitable, but also useful, Hale and Lunel (199), Krstic and Hashimoto (2016) and Michiels and Niculescu (2014).
- Habitually, time-delay leads to poor performances and often causes instability, Jankovic (2001), Hale and Lunel (1993) and Datko(1998).

Dynamic of population

Reaction diffusion equation

$$\begin{cases} x_t(s,t) & = & x_{ss}(s,t) + ax(s,t) + bx(s,t-\tau),\ t>0,\ 0<s<1 \\ x(0,t) & = & 0,\ t>0 \\ x(1,t) & = & u(t),\ t>0 \\ x(s,t) & = & \varphi(s,t),\ 0\le s\le 1,\ -\tau\le t\le 0. \end{cases} \tag{4}$$

where $a \in \mathbb{R}$, $b, \tau > 0$ and $u(t)$ is the control input.

- For population dynamics, the coefficients a and b represent the death and birth rates, respectively, and τ is the delay due to pregnancy.
- The delay term $bx(s, t-\tau)$ is expected to destabilize system (4) if b is sufficiently large. So, the control input $u(t)$ is needed to exponentially stabilize system (4).
- A constructive design method for stabilizing system (4)is used. To achieve this, we adopt the backstepping method, wherein the controlled system can be transformed into a desired exponentially stable target system.

Backstepping transformation

We use the following transformation

$$w(s,t) = x(s,t) - \int_0^s k(s,y)x(y,t)dy \qquad (5)$$

where k satisfies in $T = \{(s,y), s \in [0,1], 0 \le y \le s\}$

$$\left\{\begin{array}{lcl} a + c + 2\frac{d}{ds}k(s,s) & = & 0 \\ k(s,0) & = & 0 \\ k_{ss}(s,y) - k_{yy}(s,y) - (a+c)k(s,y) & = & 0 \end{array}\right. \qquad (6)$$

along with boundary control

$$u(t) = \int_0^1 k(1,y)x(y,t)dy \qquad (7)$$

to transform system (4) into the following target system

Target system

$$\begin{cases} w_t(s,t) &= w_{ss}(s,t) - cw(s,t) + bw(s,t-\tau),\ t>0,\ 0<s<1 \\ w(0,t) &= 0,\ t>0 \\ w(1,t) &= 0,\ t>0 \\ w(s,t) &= \psi(s,t),\ 0\le s\le 1,\ -\tau\le t\le 0. \end{cases} \tag{8}$$

where $c > 0$ is a design parameter.

Solution of (6)

By the method of successive approximations, we get

$$k(s,y) = -(a+c)y\frac{I(\sqrt{(a+c)(s^2-y^2)})}{\sqrt{(a+c)(s^2-y^2)}} \tag{9}$$

where I denotes the first-order modified Bessel function of the first kind defined by

$$I(z) = \sum_{n=0}^{+\infty} \frac{z^{2n+1}}{2^{2n+1}n!(n+1)!}$$

Hence, the control input $u(t)$ is explicit and implementable in practice.

Stability of system (8)

Let Lyapunov-Razumikhin function V be defined by

$$V(w,t) = \frac{1}{2}\|w(.,t)\|^2 = \frac{1}{2}\int_0^1 w(s,t)^2 ds.$$

Applying Wirtinger's Inequality, under assumption that $V(w(s,t+\theta)) \leq V(w(s,t))$, for all $\theta \in [-\tau, 0]$, we get

$$\dot{V} \leq -\Big(c - \frac{b^2 - \pi^2 + 4}{4}\Big)V.$$

Thus, target system (8) is rapidly exponentially stable with decay rate equal to c.

Recall that the transformation (5) is an isomorphism of Y. So, the original system (4) is also rapidly exponentially stable with the same decay rate c.

Tsunami and freak waves

Let $l > 0$, we consider the following

linear Kortweg-de Vries "KdV" equation

$$\left\{\begin{array}{lcl} u_t(x,t) & = & -u_{xxx}(x,t) - u_x(x,t) + bu(x,t-\tau), \quad t>0, \quad x \in (0,l), \\ u(0,t) & = & U(t), \quad t>0, \\ u(l,t) & = & 0, \quad t>0, \\ u_x(l,t) & = & 0, \quad t>0. \\ u(x,t) & = & \varphi(x,t), \quad t \in [-\tau,0], \quad x \in (0,l). \end{array}\right. \tag{10}$$

where $b > 0$ and $U(t)$ is the control input.
Even with $b = 0$, the corresponding open-loop system $(if U(t) = 0)$ is exponentially stable if and only if $l \notin \mathcal{N}$, where

$$\mathcal{N} := \{2\pi\sqrt{\frac{i^2+ij+j^2}{3}}; i,j \in \mathbb{N}\}.$$

$\mathcal{N}$ is called the set of critical lengths.

Theorem

For any $\lambda > 0$ (large enough), there exists $\alpha > 0$ such that

$$\|u\|_{L^2(0,l)} \leq \alpha e^{-\lambda t} \|\varphi\|_X, \forall t > -\tau.$$

Thank you for your attention