On an inverse photo-acoustic tomography problem of small absorbers with inhomogeneous sound speed

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May 9, 2022

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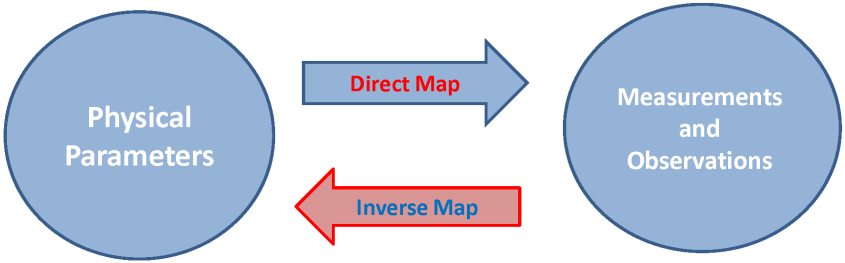
Inverse Problems



Inverse Problems

Figure 1: Plato's Allegory of the Cave

Medical Imaging

Medical Imaging

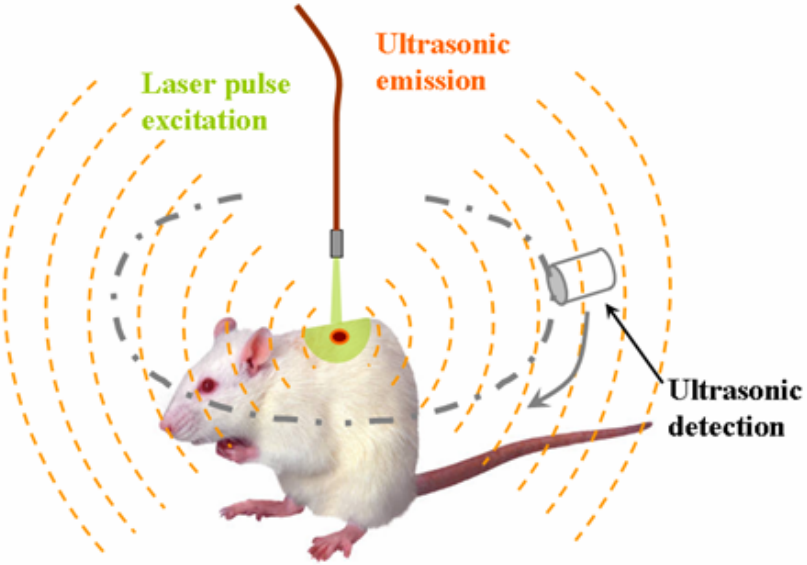
Figure 2: X-Ray

Medical Imaging

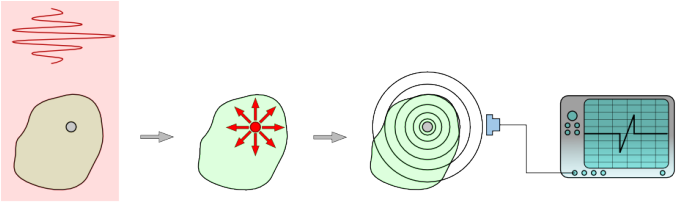
Figure 3: X-Ray

Figure 4: MRI

What is Photo-Acoustic Tomography (PAT)\(TAT)?



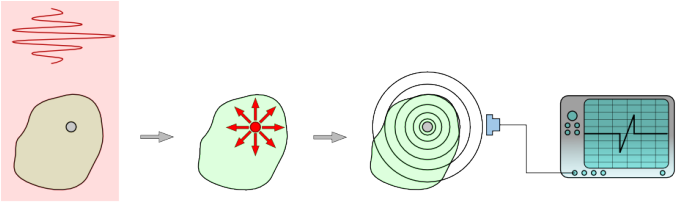
Steps of PAT-TAT



laser pulse → absorption → thermal expansion → acoustic waves → ultrasound detection → image formation

1. Sending an optical wave into the medium

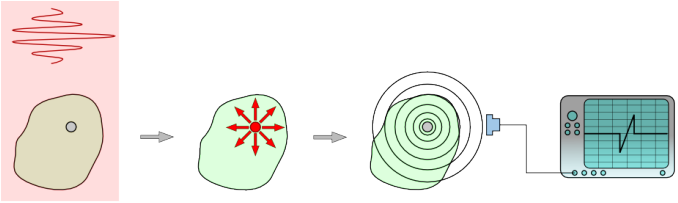
Steps of PAT-TAT



laser pulse → absorption → thermal expansion → acoustic waves → ultrasound detection → image formation

1. Sending an optical wave into the medium
2. Heating of the absorbers due to the absorption of electromagnetic energy

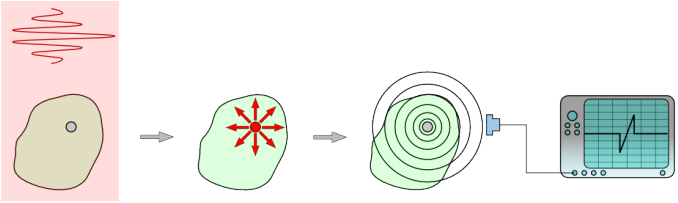
Steps of PAT-TAT



laser pulse → absorption → thermal expansion → acoustic waves → ultrasound detection → image formation

1. Sending an optical wave into the medium
2. Heating of the absorbers due to the absorption of electromagnetic energy
3. Thermal expansion of the absorbers

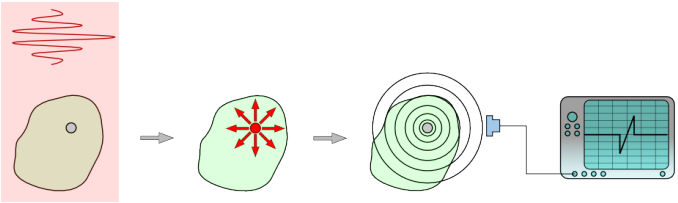
Steps of PAT-TAT



laser pulse → absorption → thermal expansion → acoustic waves → ultrasound detection → image formation

4. Generation of acoustic waves

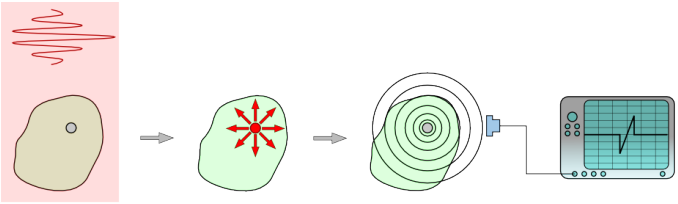
Steps of PAT-TAT



laser pulse → absorption → thermal expansion → acoustic waves → ultrasound detection → image formation

4. Generation of acoustic waves
5. Propagation of acoustic waves

Steps of PAT-TAT



laser pulse → absorption → thermal expansion → acoustic waves → ultrasound detection → image formation

4. Generation of acoustic waves
5. Propagation of acoustic waves
6. Detection of the wave at the boundary and image formation

Difference between PAT and TAT

PAT

TAT

Difference between PAT and TAT

PAT

- High frequency radiation

TAT

Difference between PAT and TAT

PAT

- High frequency radiation

TAT

- Low frequency radiation

Difference between PAT and TAT

PAT

- High frequency radiation
- Higher energy absorption

TAT

- Low frequency radiation

Difference between PAT and TAT

PAT

- High frequency radiation
- Higher energy absorption
- Low penetration due to scattering

TAT

- Low frequency radiation

Difference between PAT and TAT

PAT

- High frequency radiation
- Higher energy absorption
- Low penetration due to scattering
- Partial boundary observations

TAT

- Low frequency radiation

Difference between PAT and TAT

PAT

- High frequency radiation
- Higher energy absorption
- Low penetration due to scattering
- Partial boundary observations
- Resolution in case of variable acoustic speed

TAT

- Low frequency radiation

Difference between PAT and TAT

PAT

- High frequency radiation
- Higher energy absorption
- Low penetration due to scattering
- Partial boundary observations
- Resolution in case of variable acoustic speed

TAT

- Low frequency radiation
- Large difference in conductivity

Difference between PAT and TAT

PAT

- High frequency radiation
- Higher energy absorption
- Low penetration due to scattering
- Partial boundary observations
- Resolution in case of variable acoustic speed

TAT

- Low frequency radiation
- Large difference in conductivity
- Higher penetration of light

Difference between PAT and TAT

PAT

- High frequency radiation
- Higher energy absorption
- Low penetration due to scattering
- Partial boundary observations
- Resolution in case of variable acoustic speed

TAT

- Low frequency radiation
- Large difference in conductivity
- Higher penetration of light
- Total boundary observation

Difference between PAT and TAT

PAT

- High frequency radiation
- Higher energy absorption
- Low penetration due to scattering
- Partial boundary observations
- Resolution in case of variable acoustic speed

TAT

- Low frequency radiation
- Large difference in conductivity
- Higher penetration of light
- Total boundary observation
- Resolution only in case of constant acoustic speed

Inverse PAT problem

In PAT a high frequency radiation is delivered into the biological tissue to be imaged. The transfer of laser pulse in the biological tissue is described by the following diffusion equation.

$$\begin{aligned} -\nabla.(D(x)\nabla u) + \mu_a(x)u = &\quad 0 \ \text{ in } \Omega \\ u = &\quad f \ \text{ on } \Gamma, \end{aligned}$$

Stress Confinement condition

The complete energy is deposited almost instantaneously compared to travel time of acoustic waves.

Inverse PAT problem

Under the stress confinement condition, the acoustic pressure is assumed to be generated initially, and thus it is known to satisfy the following wave equation

$$\begin{cases} \dfrac{1}{c_s^2(x)} p_{tt}(x,t) - \Delta p(x,t) = 0 & \text{in } \Omega \times]0,T[\\ p(x,0) = p_0(x) := \beta(x)\mu_a(x)u(x) & \text{in } \Omega \\ p_t(x,0) = 0 & \text{in } \Omega \\ p(x,t) = 0 & \text{on } \Gamma \times]0,T[, \end{cases} \tag{1}$$

equipped with the partial boundary observation

$$\frac{\partial p}{\partial \nu}(x,t) := h(x,t) \quad \text{on} \quad \Sigma_T^0 := \Gamma_0 \times]0,T[,$$

where $\Gamma_0 \subset \Gamma$ satisfies the following geometric condition for some x_0 in $\mathbb{R}^3$

$$\Gamma_0 = \Gamma(x_0) = \{x \in \Gamma \text{ such that } (x - x_0).\nu > 0\}$$

Inverse PAT problem

Inverse problem

Reconstruction of the absorption coefficient μ_a from the measurement of $\frac{\partial p}{\partial \nu}$ on Σ_T^0.

Inverse PAT problem

Approaches used in literature:

The main approach used in literature for solving the PAT problem is the quantitative photo-acoustic tomography approach (qPAT) which divides the inverse problem into two main problems

1. Acoustic inversion: Reconstruction of the initial pressure $p_0(x)$ from the boundary measurements.
2. Optical inversion: Reconstruction of the absorption coefficient μ_a from the previously calculated initial pressure using the relation $p_0(x) = \beta(x)\mu_a(x)u(x)$ and the diffusion equation satisfied by u.

Inverse PAT problem

Drawbacks of the qPAT approach:

The qPAT involves many difficulties leading to the illposedness of the inverse map. These difficulties can be summarized as follows:

- Non-absorbing background medium
- Limited boundary observations
- Spatially varying Grüneisen coefficient
- Non-linear dependence of the light fluence u on the absorption coefficient

Inverse PAT problem

We assume that the absorption coefficient μ_a is defined piece-wisely,

$$\mu_a(\mathbf{x}) = \begin{cases} \mu_0(\mathbf{x}) & \text{in } \Omega \setminus \overline{\bigcup_{j=1}^{m} \omega_j} \\ \mu_j(\mathbf{x}) & \text{in } \omega_j, \end{cases}$$

where μ_j are functions belonging to the spaces $L^\infty(\omega_j)$, and $\omega_j \subset \Omega$ are small domains representing the absorbers (tumors) and defined as follows

$$\omega_j = S_j + \epsilon B_j \quad \text{for} \quad j = 1, \ldots, m,$$

$$\mu_0 = O(\epsilon^\kappa), \quad \kappa > 4.$$

Inverse PAT problem

Algebraic Algorithm

This method was first developed by El Badia and H.Doung in 2000. It consists in developing a system of algebraic relations based on the idea of the so called *Reciprocity gap functional* and Green's formula. The idea behind this algorithm is the projection of the problem onto well chosen test functions, which allows the reconstruction of the unknowns from a single boundary data.

It is based on the construction of a Hankel matrix H from the boundary observations, then the number of absorbers can be reconstructed as the rank of this matrix, and the locations of these sources are the eigenvalues of a companion matrix that is also built using the boundary measurements.

Inverse PAT problem

- **Case of constant acoustic speed**

1. Data completion
   - Reconstruction of the initial pressure $p_0(\mathbf{x})$ from the boundary observations by means of exact controllability

$$\Gamma_0 = \Gamma(\mathbf{x}_0) = \{x \in \Gamma \text{ such that } (\mathbf{x} - \mathbf{x}_0).\nu > 0\},$$

$$T > \frac{2R(\mathbf{x}_0)}{c_s},$$

   where

$$R(\mathbf{x}_0) = \sup_{\mathbf{x} \in \bar{\Omega}} |\mathbf{x} - \mathbf{x}_0|.$$

   - Data completion of $\frac{\partial p}{\partial \nu}$ can be obtained by solving (1).

Inverse PAT problem

- **Case of constant acoustic speed**

2. Resolution of the inverse problem using the algebraic algorithm. Take v to be a solution of

$$\frac{1}{c_s^2} v_{tt} - \Delta v = 0.$$

Multiplying (1) by v, and using Green's formula we get

$$\mathcal{R}(h, v) = \sum_{j=1}^{N} \int_{w_j} \frac{1}{c_s^2} \beta(\mathbf{x}) \mu_j(\mathbf{x}) u(\mathbf{x}) v_t(\mathbf{x}, 0) d\mathbf{x} + O(\epsilon^\kappa),$$

where

$$\mathcal{R}(h, v) := \int_\Omega \frac{1}{c_s^2} (p(\mathbf{x}, T) v_t(\mathbf{x}, T) - p_t(\mathbf{x}, T) v(\mathbf{x}, T)) d\mathbf{x} + \int_{\Sigma_T} v(\xi, t) h(\xi, t) d\xi dt.$$

Inverse PAT problem

- **Case of constant acoustic speed**

2. Resolution of the inverse problem using the algebraic algorithm. Take v to be a solution of

$$\frac{1}{c_s^2} v_{tt} - \Delta v = 0.$$

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$$\mathcal{R}(h, v) := \int_\Omega \frac{1}{c_s^2} (p(\mathbf{x}, T) v_t(\mathbf{x}, T) - p_t(\mathbf{x}, T) v(\mathbf{x}, T)) d\mathbf{x} + \int_{\Sigma_T} v(\xi, t) h(\xi, t) d\xi dt.$$

$$v(., T) = v_t(., T) = 0$$

$$v_t(., 0) \neq 0.$$

Inverse PAT problem

- **Case of constant acoustic speed**

We define for $n \in \mathbb{N}$

$$v_n^a(x,y,z,t) = \rho(z,t)(x+iy)^n,$$

where ρ is a solution of the one dimensional wave equation

$$\begin{cases} \dfrac{1}{c_s^2}\rho_{tt}(z,t) - \rho_{zz}(z,t) = 0 & \text{in } (\gamma_1,\gamma_2)\times(0,T) \\ \rho(z,0) = 0 \quad \rho_t(z,0) = 1 & \text{in } (\gamma_1,\gamma_2) \\ \rho(\gamma_1,t) = 0 \; \rho(\gamma_2,t) = w(t) & \text{for } t \in (0,T). \end{cases}$$

Taking $T > 2\frac{\mathrm{diam}\ (\Omega)}{c_s}$, then by exact controllability, we can find $w \in H_0^1(0,T)$, such that ρ satisfies

$$\rho(.,T) = \rho_t(.,T) = 0.$$

Inverse PAT problem

- **Case of constant acoustic speed**

Substituting v_n^a in the previous formula we get

$$\mathcal{R}(f, v_n^a) = \sum_{j=1}^{N} \sum_{k=0}^{n} \nu_j^{k,a} \binom{n}{k} (P_j^a)^{n-k} + O(\epsilon^\kappa), \quad \text{for all} \quad n \in \mathbb{N}$$

where

$$\nu_j^{k,a} = \epsilon^{3+k} \int_{B_j} \frac{1}{c_s^2} \lambda_j (S_j + \epsilon\tau)(\tau_1 + i\tau_2)^k d\tau.$$

$$\lambda_j(\mathbf{x}) = \beta(\mathbf{x}) \mu_j(\mathbf{x}) u(\mathbf{x}).$$

Inverse PAT problem

- **Case of constant acoustic speed**

Obtained Results:

- Reconstruction of the number of absorbers N using a Hankel matrix H^r;
- Reconstruction of the projections P_j^r of the centers using a companion matrix;
- Reconstruction of the coefficients $\nu_j^{k,r}$ as a solution of a linear system via the Hankel matrix H^r;
- Reconstruction of the final locations S_j by solving a similar linear system and using the coefficients $\nu_j^{k,r}$;
- Hölder stability estimates for the reconstruction of the centers S_j;

Inverse PAT problem

- **Case of variable acoustic speed**

Test functions of the form $\rho(z,t)(x+iy)^n$ are not applicable in this case, for this purpose, we opt to reconstruct the final pressure $p(.,T)$.

1. Data completion
   - Reconstruction of the initial pressure $p_0(\mathbf{x})$ from the boundary observations by means of exact controllability for variable speed

$$s_1 = \|\nabla c_s\|_\infty < \frac{c_{\min}}{2R(\mathbf{x}_0)},$$

$$T > \frac{2s_1 R(\mathbf{x}_0)}{c_{\min} - 2s_1 R(\mathbf{x}_0)}.$$

   - Data completion of $\frac{\partial p}{\partial \nu}$ and $p(.,T)$ can be obtained by solving (1).
2. Resolution of the inverse problem using the algebraic algorithm and the test functions

$$v_n^a(x,y,z,t) = (t-T)(x+iy)^n.$$

Inverse PAT problem

- **Case of variable acoustic speed**

Following the algebraic algorithm we can finally obtain

- The number of the absorbers
- The locations of the centers
- Hölder stability estimate for the reconstruction of the centers.

Inverse moving point source problem for the wave equation

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Further Work

$$\begin{cases} \dfrac{1}{c^2}\phi_{tt} - \Delta\phi = \lambda\delta(\mathbf{x} - b(t)) & \text{in } \mathbb{R}^3 \times (0,T) \\ \phi(\mathbf{x},0) = \phi_t(\mathbf{x},0) = 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{2}$$

where $T > 0$ is a fixed time, $c > 0$ is the speed of the wave, $\lambda > 0$ is the intensity, and $b \in C^2([0,T];\mathbb{R}^3)$ is the position of the point source confined within a bounded domain $D \subset \mathbb{R}^3$. Let Ω be a smooth bounded domain satisfying $\overline{D} \subset \Omega$ with boundary Γ. Notice that the trajectory of the point source remains away from Γ.

Goal: Reconstruct the trajectory followed by the source term b by measuring ϕ on six well chosen points on the observation surface Γ.


Thank
You