Exact boundary controllability of the linear Biharmonic Schrödinger equation with variable coefficients

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The control problem

We consider the following control system

$$\begin{cases} i\rho(x)\partial_t y = -\partial_x^2\left(\sigma(x)\partial_x^2 y\right) + \partial_x\left(q(x)\partial_x y\right), \ (t,x) \in (0,T)\times(0,\ell), \\ y(t,0) = \partial_x y(t,0) = y(t,\ell) = 0, \ \partial_x y(t,\ell) = f(t), \quad t \in (0,T), \\ y(0,x) = y_0(x), \quad x \in (0,\ell), \end{cases} \tag{1}$$

where f is a control that acts at the right end $x = \ell > 0$, and the function y_0 is the initial condition.

The main assumptions

- We assume that $\rho(x) > 0, \ \sigma(x) > 0$ and $q(x) \geq 0$, such that

$$\rho, \sigma \in H^2(0,\ell), \ q \in H^1(0,\ell). \tag{2}$$

- The control $f \in L^2(0,T)$.

Exact controllability question

System (1) is said to be exactly controllable at time $T > 0$ if, for any given initial state y_0, can we find a control $f \in L^2(0,T)$ such that the corresponding solution $y = y(t,x)$ satisfies $y(T,.) = 0$.

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The Biharmonic Schrödinger equation

The fourth-order cubic nonlinear Schrödinger equation or the so-called biharmonic cubic non-linear Schrödinger equation reads as follows

$$i\partial_t y + \sigma\partial_x^4 y - q\partial_x^2 y - \mu|y|^2 y = 0, \tag{3}$$

where y is a complex-valued function, σ, q and μ are real constants.

- This equation has been modeled by Karpman [1], and Karpman & Shagalov [2] in order to describe the propagation of intense laser beams in a bulk medium with Kerr nonlinearity when small fourth-order dispersion is taken into account.

[1] **Karpman [Phys. Rev. E, 1995]**.

[2] **Karpman and Shagalov [Physica D : Nonl. Phen., 2000]**.

- The biharmonic Schrödinger equation (3) has various applications in several fields of physics, such as nonlinear optics, plasma physics, superconductivity and quantum mechanics.

Controllability and stabilization of Equation (3) for $\partial_x^4 y = \mu = 0$.

- Notice that in the case where $\partial_x^4 y = 0 = \mu$ and $q > 0$, Equation (3) collapse to the standard second order Schrödinger equation

$$i\partial_t y - q\partial_x^2 y = 0.$$

Machtyngier, [SIAM J. Cont. Optim., 1994].

Zuazua, [Quantum Control : Math. and Numer. Chall, 2003].

Ammari, Mercier and Régnier, [Journal of Diff. Equat., 2015].

Hansen, [Amer. Cont. Conf., 2017].

Ammari, Choulli and Robbiano [Journal of Diff. Equat, 2019].

Ammari and Duca [Inter. Jour. of Cont., 2021].

Ammari and Sabri [Anal. Math. Phys., 2022].

Controllability and stabilization of Equation (3) for $\partial_x^2 y = 0 = \mu$.

- In the case where $\partial_x^2 y = 0 = \mu$ and $\sigma = 1$, Equation (3) becomes,

$$i\partial_t y + \partial_x^4 y = 0.$$

Zheng and Chen [Chin. Ann. Math., 2012].

Wen, Chai and Guo, [SIAM J. Control Optim., 2014].

Wen, Chai and Guo, [Math. Cont. Signa. Syst., 2016].

Aksas and Rebiai [J. Math. Anal. Appl., 2017].

- Controllability and stabilization of Equation (3) for $\sigma = q = \mu = 1$, i.e.,

$$i\partial_t y + \partial_x^4 y - \partial_x^2 y - |y|^2 y = 0.$$

Filho and Cavalcante [Appl. Math. Optim., 2021].

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The spectral problem

- We consider the following spectral problem,

$$\begin{cases} (\sigma(x)\phi'')'' - (q(x)\phi')' = \lambda\rho(x)\phi, & x \in (0,\ell), \\ \phi(0) = \phi'(0) = \phi(\ell) = \phi'(\ell) = 0. \end{cases} \tag{4}$$

- Let $L^2_\rho(0,\ell)$ be the weighted Lebesgue space of all complex-valued functions defined on $(0,\ell)$, which is equipped with the inner product

$$\langle y, \bar{z}\rangle_{L^2_\rho(0,\ell)} = \int_0^\ell y(x)\bar{z}(x)\rho(x)dx, \ \forall\ y,z \in L^2_\rho(0,\ell). \tag{5}$$

- In the sequel, we introduce the operator $\mathcal{A}$ defined in $L^2_\rho(0,\ell)$ by setting :

$$\mathcal{A}y = \rho^{-1}\left((\sigma y'')'' - (qy')'\right),\ \mathcal{D}(\mathcal{A}) = H^4(0,\ell) \cap H^2_0(0,\ell).$$

Lemma 1

The linear operator $\mathcal{A}$ is positive and self-adjoint such that $\mathcal{A}^{-1}$ is compact. Moreover, the spectrum of $\mathcal{A}$ is discrete and consists of a sequence of positive eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ tending to $+\infty$. The corresponding eigenfunctions $(\Phi_n)_{n\in\mathbb{N}^*}$ can be chosen to form an orthonormal basis in $L^2_\rho(0,\ell)$.

Simplicity of $(\lambda_n)_{n\in\mathbb{N}^*}$ and non-vanishing of $(\phi_n''(\ell))_{n\in\mathbb{N}^*}$

Theorem [Ammari-B, 2021]

All the eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ of the spectral problem (4) are simple such that :

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots \underset{n\to+\infty}{\longrightarrow} +\infty.$$

Moreover, the corresponding eigenfunctions $(\Phi_n)_{n\in\mathbb{N}^*}$ satisfy

$$\Phi_n''(\ell) \neq 0 \quad \forall n \in \mathbb{N}^*. \tag{6}$$

Key ideas

The proof uses an extension of Leighton & Neahri Lemma [Trans. AMS, 1957].

Lemma

Let u be a nontrivial solution to the linear fourth order differential equation defined on the segment $[0,\ell]$:

$$(\sigma(x)u'')'' - (q(x)u')' - \rho(x)u = 0,$$

where the functions $\rho(x) > 0$, $\sigma(x) > 0$ and $q(x) \geq 0$. If u, u', u'' and $\mathcal{T}u = (\sigma(x)u'')' - q(x)u'$ are nonnegative at $x = 0$ (but not all zero), then they are positive for all $x > 0$. If $u, -u', u''$ and $(-\mathcal{T}u)$ are nonnegative at $x = \ell$ (but not all zero), then they are positive for all $x < \ell$.

Asymptotic expansions of the spectral gaps and the eigenfunctions

Theorem [Ammari-B, 2021]

(a) The eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ of the associated spectral problem (4) satisfy the following asymptotic

$$\sqrt[4]{\lambda_n} := \mu_n = \frac{\pi}{\gamma}\left(n - \frac{1}{2}\right) + \mathcal{O}\left(\frac{1}{\exp(n)}\right), \ \gamma := \int_0^\ell \left(\frac{\rho(t)}{\sigma(t)}\right)^{\frac{1}{4}} dt. \tag{7}$$

Moreover, one has

$$|\lambda_{n+1} - \lambda_n| = \frac{4\pi^4}{\gamma^4} n^3 + \mathcal{O}\left(n^2\right). \tag{8}$$

(b) Assume that the eigenfunctions $(\Phi_n)_{n\in\mathbb{N}^*}$ of the spectral problem (4) are normalized in the sense that $\lim_{n\to\infty} \|\Phi_n\|_{L^2_\rho(0,\ell)} = 1$. Then, we have the following asymptotic estimates

$$\Phi_n(x) = \gamma^{-\frac{1}{2}} \zeta(x) \left(\sin(\mu_n \mathcal{X}) - \cos(\mu_n \mathcal{X})\right), \quad \text{as } n \to \infty, \tag{9}$$

where $\zeta(x) := \left([\rho(x)]^{\frac{3}{4}} [\sigma(x)]^{\frac{1}{4}}\right)^{-\frac{1}{2}}$ and $\mathcal{X} := \int_0^x \left(\frac{\rho(t)}{\sigma(t)}\right)^{\frac{1}{4}} dt$. Furthermore, one has

$$\lim_{n\to\infty} \frac{|\Phi_n''(\ell)|}{\sqrt{\lambda_n}} = \zeta(\ell) \left(\frac{\rho(\ell)}{\gamma\sigma(\ell)}\right)^{\frac{1}{2}}. \tag{10}$$

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Well-posedness of the adjoint system

- As System (1) is a self-adjoint one, we are reduced to the same system, without control. Therefore, we consider the following system

$$\begin{cases} i\rho(x)\partial_t z = -\partial_x^2\left(\sigma(x)\partial_x^2 z\right) + \partial_x(q(x)\partial_x z), \ (t,x) \in (0,T)\times(0,\ell), \\ z(t,0) = \partial_x z(t,0) = z(t,\ell) = \partial_x z(t,\ell) = 0, \ t \in (0,T), \\ z(0,x) = z_0, \ x \in (0,\ell). \end{cases} \tag{11}$$

- From the previous section, the operator $\mathcal{A}$ generates a scale of interpolation spaces $\mathcal{H}_\theta$, $\theta \in \mathbb{R}$,

$$\mathcal{H}_\theta = \left\{ u(x) = \sum_{n\in\mathbb{N}^*} c_n \Phi_n(x) \ : \ \|u\|_\theta^2 = \sum_{n\in\mathbb{N}^*} \lambda_n^{2\theta} |c_n|^2 < \infty \right\}. \tag{12}$$

- In particular, $\mathcal{H}_0 = L^2_\rho(0,\ell)$ and $\mathcal{H}_{1/2} = H_0^2(0,\ell)$.

Lemma

Let $\theta \in \mathbb{R}$ and $z_0 \in \mathcal{H}_\theta$. Then Problem (11) has a unique solution $z \in C([0,T],\mathcal{H}_\theta)$, which is written in Fourier series as

$$z(t,x) = \sum_{n\in\mathbb{N}^*} c_n e^{i\lambda_n t}\Phi_n(x), \tag{13}$$

where $z_0 = \sum_{n\in\mathbb{N}^*} c_n \Phi_n$. Moreover, the energy $\mathcal{E}_\theta(t)$ of System (11) is conserved along the time.

Observability

Proposition [Ammari-B, 2021]

Let $T > 0$ and $z_0 \in H_0^2(0, \ell)$. Then, there exits a positive constant $C_T > 0$, depending on T, such that

$$C_T^{-1} \|z_0\|^2_{H_0^2(0,\ell)} \leq \int_0^T |\partial_x^2 z(t,\ell)|^2 dt \leq C_T \|z_0\|^2_{H_0^2(0,\ell)}, \tag{14}$$

where z is the solution of Problem (11).

Key ideas

Beurling's Theorem

Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a strictly increasing sequence satisfying for some $\delta > 0$ the condition

$$|\lambda_{n+1} - \lambda_n| > \delta, \ \forall \ n \in \mathbb{Z}.$$

Then, for any $T > 2\pi D^+ (\lambda_n)$, the family $\left(e^{i\lambda_n t}\right)_{n\in\mathbb{Z}}$ forms a Riesz basis in $L^2(0, T)$, that is

$$C_T^{-1} \sum_{n\in\mathbb{Z}} |\widehat{c}_n|^2 \leq \int_0^T |\sum_{n\in\mathbb{N}^*} \widehat{c}_n e^{i\lambda_n t}|^2 dt \leq C_T \sum_{n\in\mathbb{Z}} |\widehat{c}_n|^2, \ C_T > 0,$$

where $D^+ (\lambda_n)$ is the Beurling upper density of the sequence $(\lambda_n)_{n\in\mathbb{N}^*}$.

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Main controllability result

Well-posedness

Let $T > 0$, and $f \in L^2(0,T)$. Then for any $y_0 \in H^{-2}(0,\ell)$, there exists a unique weak solution of System (1) in the sense of transposition, i.e.,

$$i\langle \bar{y}(T,x), z(T,x)\rangle_{H^{-2},H_0^2} = i\langle \bar{y}_0, z_0\rangle_{H^{-2},H_0^2} + \sigma(\ell)\int_0^T \bar{f}(t)\partial_x^2 z\,(t,\ell)\,dt, \quad (15)$$

satisfying $y \in C\left([0,T]; H^{-2}(0,\ell)\right)$. Moreover, there exists a constant $C(T) > 0$ such that

$$\|y\|_{L^\infty\left([0,T];H^{-2}(0,\ell)\right)} \le C(T)\left(\|y_0\|_{H^{-2}(0,\ell)} + \|f\|_{L^2(0,T)}\right). \quad (16)$$

Theorem [Ammari-B, 2021]

Assume that the coefficients $\rho > 0$, $\sigma > 0$, $q \ge 0$ and satisfy relation (2). Given $T > 0$ and $y_0 \in H^{-2}(0,\ell)$, there exists a control $f \in L^2(0,T)$ such that the solution y of the control problem (1) satisfies

$$y(T,x) = 0, \quad x \in [0,\ell].$$

This Theorem follows from the Lions' HUM.

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Lack of Exact Controllability

- Question : What happens if $q(x) < 0$?
- We consider the following control system

$$\begin{cases} i\partial_t y(t,x) = -\partial_x^4 y(t,x) + q\ \partial_x^2 y(t,x), & (t,x) \in (0,T) \times (0,\ell), \\ y(t,0) = y(t,\ell) = \partial_x^2 y(t,\ell) = 0, \ \partial_x^2 y(t,0) = f(t), & t \in (0,T), \\ y(0,x) = y_0(x), \ x \in (0,\ell), \end{cases} \tag{17}$$

where the parameter $q < 0$, $f \in L^2(0,T)$ is a control that acts at the left end $x = 0$, and the function y_0 is the initial condition.

- Let

$$\Gamma^* = \left\{ -\frac{\pi^2}{\ell^2}\left(p^2 + k^2\right) \ : \ p, k \in \mathbb{N}^*, \ 1 \leq p < k \right\}. \tag{18}$$

- Response : Lack of exact controllability, in the case where $q \in \Gamma^*$.
- We consider, the associated spectral problem

$$\begin{cases} \phi'''' - q\ \phi'' = \lambda\phi, & x \in (0,\ell), \\ \phi(0) = \phi''(0) = \phi(\ell) = \phi''(\ell) = 0. \end{cases} \tag{19}$$

Spectral Analysis

Theorem [Ammari-B, 2022]

(a) All the eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ of Problem (19) are real and tending to $+\infty$:

$$-\frac{q^2}{4} \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots \underset{n\to+\infty}{\to} +\infty.$$

(b) All the positives eigenvalues of Problem (19) are algebraically simple and satisfy

$$\lambda_n = \frac{n^2\pi^2}{\ell^2}\left(\frac{n^2\pi^2}{\ell^2} + q\right) \text{ for } n > \ell\pi^{-1}\sqrt{|q|}. \tag{20}$$

(c) $\lambda = 0$ is a simple eigenvalue of Problem (19) if and only if

$$q = -\frac{p^2\pi^2}{\ell^2},\ p \in \mathbb{N}^*. \tag{21}$$

(d) All the negatives eigenvalues of Problem (19) satisfy

$$\lambda_n = \frac{n^2\pi^2}{\ell^2}\left(\frac{n^2\pi^2}{\ell^2} + q\right) \text{ for } n < \ell\pi^{-1}\sqrt{|q|}. \tag{22}$$

Moreover, they are algebraically simple, if and only if, $q \notin \Gamma^*$.

Main controllability result

Theorem [Ammari-B, 2022]

Given $T > 0$, $y_0 \in H^{-1}(0, \ell)$.
(a) If $q \notin \Gamma^*$, then, System (17) is exactly controllable in time $T > 0$.
(b) If $q \in \Gamma^*$, then, System (17) is not exactly controllable in time $T > 0$.

Sketch of proof The control problem is reduced to the obtention of suitable observability inequalities for the adjoint system (17) with $f \equiv 0$, that is,

$$C_T^{-1}\|z_0\|^2_{H_0^1(0,\ell)} \leq \int_0^T |\partial_x z(t,0)|^2 dt \leq C_T\|z_0\|^2_{H_0^1(0,\ell)}, \; C_T > 0. \qquad (23)$$

Let $q \in \Gamma^*$ and let z be the solution of the adjoint system with initial data

$$z_0 := \Phi_n(x) = \sqrt{\frac{\ell}{2}}\phi(x, \lambda_n), \quad \text{for some } n < \ell\pi^{-1}\sqrt{|q|}.$$

Then,

$$z(t, x) = \sqrt{\frac{\ell}{2}}e^{i\lambda_n t}\phi(x, \lambda_n), \text{ for some } n < \ell\pi^{-1}\sqrt{|q|}.$$

Therefore, $\partial_x z(t, 0) = 0$. Consequently, the right hand side of the first inequality in (23) is zero, while, the left hand side is not zero. Thus, the first inequality in (23) cannot be valid.

K. Ammari and H. Bouzidi, *Exact boundary controllability of the linear Biharmonic Schrödinger equation with variable coefficients*, https://arxiv.org/abs/2112.15196, in revision.

K. Ammari and H. Bouzidi, *Positive and negative exact boundary controllability results for the linear Biharmonic Schrödinger equation*, https://arxiv.org/abs/2204.11963, submitted.

Thank you for your attention !