Spectral analysis and numerical results in non-simple elastic memory plate

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The linear anti-plane shear equations of nonsimple viscoelasticity corresponding to the isotropic homogeneous case.

$$u_{tt}(x,t) - \alpha\Delta u(x,t) + \gamma\Delta^2 u(x,t) + \int_0^\infty \Big(m_1(s)\Delta u(x,t-s) - m_2(s)\Delta^2 u(x,t-s)\Big)ds = 0$$

in $\Omega \times [0,\infty)$, where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Where u represents the vertical displacement of the plate, α and γ are the material coefficients and m_1 and m_2 are the memory kernels.

- **Dirichlet boundary conditions**

$$u(x,t) = \Delta u(x,t) = 0, \quad x \in \partial\Omega, \qquad t \geq 0, \tag{1}$$

- **Initial conditions**

$$u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), \quad x \in \Omega, \tag{2}$$

Hypothesis

Concerning the memory kernels $m_i, \ i = 1, 2$, we assume that

$(\mathbb{H}_1)$ $m_i \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$,

$(\mathbb{H}_2)$ $m_i(s) \geq 0 \quad \forall s \in \mathbb{R}^+$,

$(\mathbb{H}_3)$ $m_i'(s) \leq 0 \quad \forall s \in \mathbb{R}^+$,

$(\mathbb{H}_4)$ $m_i'(s) + \delta_i m_i(s) \leq 0$ for some $\delta_i > 0, \ \forall s \in \mathbb{R}^+$,

$(\mathbb{H}_5)$ $m_i(0) = \int_0^\infty m_i(s) ds := m_i^0 > 0$.

Introducing the new variable $v = u_t$, setting $U = (u, v, \zeta)$, the problem can be written as a linear evolution equation in $\mathscr{H}$ of the form

$$\frac{dU}{dt} = \mathcal{A}U, \qquad U(0) = U_0, \tag{3}$$

where $U(0) = (u_0, u_1, \zeta_0) \in \mathscr{H}$ and $\mathcal{A} : \mathscr{D}(\mathcal{A}) \subset \mathscr{H} \to \mathscr{H}$ is the linear operator defined by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ \zeta \end{pmatrix} = \begin{pmatrix} v \\ -\alpha Au - \gamma A^2 u - \int_0^\infty m_1(s)A\zeta(s)ds - \int_0^\infty m_2(s)A^2\zeta(s)ds \\ v - \partial_s \zeta \end{pmatrix} \tag{4}$$

with the domain

$$\mathscr{D}(\mathcal{A}) = \left\{ U \in \mathscr{H} \;\middle|\; \begin{array}{c} v \in V_2 \\ \alpha Au + \gamma A^2 u + \int_0^\infty m_1(s)A\zeta(s)ds + \int_0^\infty m_2(s)A^2\zeta(s)ds \in V_0 \\ \partial_s\zeta \in \mathcal{W}, \;\; \zeta(0) = 0 \end{array} \right\}.$$

Lemma

Suppose that conditions $(\mathbb{H}_1) - (\mathbb{H}_5)$ *hold, then the operator* $\mathcal{A}$ *generates a semigroup of contractions in* $\mathscr{H}$.

Proof : We start by showing that operator $\mathcal{A}$ is dissipative. Let $U = (u, v, \zeta)$ be in $\mathscr{D}(\mathcal{A})$. From (4), the divergence theorem and the boundary conditions, it is quite easy to check that

$$
\begin{aligned}
< \mathcal{A}U, U >_{\mathscr{H}} &= - < \partial_s \zeta, \zeta >_{\mathcal{W}} \\
&= -\frac{1}{2}\int_0^\infty m_1(\sigma)\frac{d}{d\sigma}\|\zeta(\sigma)\|_1^2 d\sigma - \frac{1}{2}\int_0^\infty m_2(\sigma)\frac{d}{d\sigma}\|\zeta(\sigma)\|_2^2 d\sigma \\
&= \frac{1}{2}\int_0^\infty m_1'(\sigma)\|\zeta(\sigma)\|_1^2 d\sigma + \frac{1}{2}\int_0^\infty m_2'(\sigma)\|\zeta(\sigma)\|_2^2 d\sigma \\
&\leq -\frac{\kappa_1}{2}\|\zeta\|_{\mathcal{M}_1}^2 - \frac{\kappa_2}{2}\|\zeta\|_{\mathcal{M}_2}^2 < 0.
\end{aligned}
\tag{5}
$$

This proves that $\mathcal{A}$ is a dissipative operator.

- Secondly, we show that the operator $\mathcal{A}$ has the property that Range $(\mathcal{I} - \mathcal{A}) = \mathscr{H}$, the operator $\mathcal{A}$ is maximal dissipative in $\mathscr{H}$
- Since $\mathscr{D}(\mathcal{A})$ is densely defined in $\mathscr{H}$, from the Lumer-Phillips corollary to the Hille-Yosida theorem [1], we conclude that the operator $\mathcal{A}$ generates a semigroup of contractions in $\mathscr{H}$.

Now an application of the theory of semigroups (see Pazy [1]) gives

Theorem

Suppose that conditions $(\mathbb{H}_1) - (\mathbb{H}_5)$ hold, the operator $\mathcal{A}$ generates a C_0-semigroup $T(t) = e^{t\mathcal{A}}$ on $\mathscr{H}$. Hence, the system (3) is well-posed, i.e., for any $U_0 \in \mathscr{H}$, the system (3) has a unique weak solution $U(t) = e^{t\mathcal{A}}U_0$. Furthermore, if $U_0 \in \mathscr{D}(\mathcal{A})$, $U(t) = e^{t\mathcal{A}}U_0$, becomes the classic solution to (3).

A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, *Springer-Verlag, Berlin* (1983).

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Spectral analysis of the problem

To simplify the analysis in the next sections, we consider that the memory kernels are given by the following form

$$m_1(\tau) = m_2(\tau) = \beta e^{-\eta\tau}, \qquad \beta,\ \eta > 0, \tag{6}$$

Consider the positive operators A and A^2 on $X = L^2(\Omega)$ defined by $A\phi = -\Delta\phi$ and $A^2\phi = \Delta^2\phi$ with Dirichlet boundary conditions and with the domains $\mathscr{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and $\mathscr{D}(A^2) = H^4(\Omega)$. The operator A has the following very well-known properties.

(a) The spectrum of $A = -\Delta$ consists of only eigenvalues

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \infty, \tag{7}$$

each one with multiplicity one.

(b) The eigenfunctions of A with Dirichlet boundary conditions are real analytic functions.

(c) For all $\mathrm{x} \in \mathscr{D}(A)$ we have

$$A\mathrm{x} = \sum_{n=1}^{\infty} \lambda_n \langle \mathrm{x}, \phi_n \rangle \phi_n = \sum_{n=1}^{\infty} \lambda_n E_n \mathrm{x},$$

$\langle \cdot, \cdot \rangle$ is the inner product in $X = L^2(\Omega)$ and $E_n \mathrm{x} = \langle \mathrm{x}, \phi_n \rangle \phi_n$. So $\{E_n\}$ is a complete family of orthogonal projections in X and $\mathrm{x} = \sum_{n=1}^{\infty} E_n \mathrm{x},\ \mathrm{x} \in X$.

(d) The fractional powered spaces X^r are given by

$$X^r = \mathscr{D}(A^r) = \left\{ \mathrm{x} \in X,\ \sum_{n=1}^{\infty} (\lambda_n)^{2r} \|E_n \mathrm{x}\|^2 < \infty \right\},\ r \geq 0$$

$$\begin{cases} \|\mathrm{x}\|_{X^r} = \|A^r \mathrm{x}\| = \left\{ \sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n \mathrm{x}\|^2 \right\}^{1/2},\ \mathrm{x} \in X^r \\ A^r \mathrm{x} = \sum_{n=1}^{\infty} \lambda_n^r E_n \mathrm{x}. \end{cases} \tag{8}$$

The problem (1) can be written as a linear evolution equation in the Hilbert space $\mathscr{H}_1 = X^1 \times X \times X$ of the form

$$z' = \mathcal{A}z, \qquad z(0) = z^0, \tag{9}$$

where $z = (u, u_t, \varphi)$, and $\mathcal{A} : \mathscr{D}(\mathcal{A}) \subset \mathscr{H}_1 \to \mathscr{H}_1$ is given by

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 \\ -(\alpha A + \gamma A^2) & 0 & \beta(A + A^2) \\ I & 0 & -\eta I \end{pmatrix} \tag{10}$$

Computing $\mathcal{A}z$ yields

$$\mathcal{A}z = \begin{pmatrix} \sum_{n=1}^{\infty} E_n z_1 \\ -\sum_{n=1}^{\infty} (\alpha\lambda_n + \gamma\lambda_n^2) E_n z_0 + \beta \sum_{n=1}^{\infty} (\lambda_n + \lambda_n^2) E_n z_2 \\ \sum_{n=1}^{\infty} E_n z_0 - \eta \sum_{n=1}^{\infty} E_n z_2 \end{pmatrix}$$
$$= \sum_{n=1}^{\infty} \begin{pmatrix} 0 & 1 & 0 \\ -(\alpha\lambda_n + \gamma\lambda_n^2) & 0 & \beta(\lambda_n + \lambda_n^2) \\ 1 & 0 & -\eta \end{pmatrix} \begin{pmatrix} E_n & 0 & 0 \\ 0 & E_n & 0 \\ 0 & 0 & E_n \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix}$$
$$= \sum_{n=1}^{\infty} A_n P_n z, \ z \in \mathscr{D}(\mathcal{A}), \tag{11}$$

where $\{P_n\}_{n\geq 1}$ is a complete family of orthogonal projections in the Hilbert space $\mathscr{H}_1$

$$P_n = \begin{pmatrix} E_n & 0 & 0 \\ 0 & E_n & 0 \\ 0 & 0 & E_n \end{pmatrix}, \quad P_iP_j = \begin{cases} P_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \qquad \sum_{n\geq 1} P_n = I, \tag{12}$$

and

$$A_n = \begin{pmatrix} 0 & 1 & 0 \\ -(\alpha\lambda_n + \gamma\lambda_n^2) & 0 & \beta(\lambda_n + \lambda_n^2) \\ 1 & 0 & -\eta \end{pmatrix}, \; n \geq 1. \tag{13}$$

The characteristic equation of A_n is given by

$$\sigma^3 + \eta\sigma^2 + c_n\sigma + \eta c_n(\alpha, \gamma) - \beta c_{n,1} = 0, \tag{14}$$

where

$$c_n = c_n(\alpha, \gamma) = \alpha\lambda_n + \gamma\lambda_n^2, \quad c_{n,1} = c_n(1, 1) = \lambda_n + \lambda_n^2. \tag{15}$$

The roots of (14), $\sigma_i(n),\ i = 0, 1, 2$, are given by the following

Proposition : Let suppose that condition

$$\eta > \frac{9}{8}\beta \max\left\{\frac{1}{\gamma}, \frac{1+\lambda_1}{\alpha+\lambda_1\gamma}\right\} \quad (16)$$

holds and

$$\delta_0(n) = \eta^2 - 3c_n, \quad \delta_1(n) = 2\eta^3 + 9(2\eta c_n - 3\beta c_{n,1}),$$
$$C(n) = \sqrt[3]{\frac{1}{2}\left(\delta_1(n) + \sqrt{\delta_1^2(n) - 4\delta_0^3(n)}\right)}, \quad (17)$$

where $\sqrt{.}$ and $\sqrt[3]{.}$ stand for the main branch of complex square and cubic roots. The spectrum of (14) consists of a sequence of conjugate pairs $\{\sigma_1(n)\}_{n=1}^{\infty}$, $\{\sigma_2(n) = \overline{\sigma_1}(n)\}_{n=1}^{\infty}$ and a real sequence $\{\sigma_0(n)\}_{n=1}^{\infty}$ where

$$\sigma_i(n) = -\frac{1}{3}\left(\eta + C(n)e^{\frac{2i\pi}{3}\underline{\mathrm{i}}} + \frac{\delta_0(n)}{C(n)}e^{\frac{-2i\pi}{3}\underline{\mathrm{i}}}\right), \quad i = 0, 1, 2, \quad n \geq 1, \quad (18)$$

where $\underline{\mathrm{i}}$ is the imaginary unit ($\underline{\mathrm{i}}^2 = -1$). Moreover, we have

$$\Re e\, \sigma_i(n) < 0 \text{ for all } i = 0, 1, 2, \quad n \geq 1. \quad (19)$$

Lemma

We suppose that condition (16) holds. The asymptotic expressions of the eigenvalues $\sigma_i(n)$, $i = 0, 1, 2,\ n \geq 1$, *of* (14) *are given by*

$$\sigma_0(n) = \frac{\beta}{\gamma} - \eta + O(\lambda_n^{-1}) \quad and \quad \sigma_1(n) = -\frac{\beta}{2\gamma} - \underline{i}\left(\frac{\alpha}{2\sqrt{\gamma}} + \sqrt{\gamma}\lambda_n\right) + O(\lambda_n^{-1}),$$

$$as \quad n \to \infty.$$

Remark : We see that under the condition (16), we have $\sigma_i(n) < 0$ for all $i = 0, 1, 2$ and $n \geq 1$, which agrees with (19).
In the following we use Lemma 2 to show that $\Re e\, \sigma_i(n)$ is strictly monotone.

Lemma

We suppose that condition (16) holds. Then, $\{\Re e\, \sigma_i(n)\}_{n\geq 1}$ is strictly monotone, more precisely $\{\sigma_0(n)\}_{n\geq 1}$ is strictly increasing and $\{\Re e\, \sigma_1(n)\}_{n\geq 1} = \{\Re e\, \sigma_2(n)\}_{n\geq 1}$ is strictly decreasing with

$$\begin{aligned} \sigma_0(1) &< \cdots < \sigma_0(n-1) < \sigma_0(n) \\ \sigma_0(n) &< \Re e\, \sigma_1(n) < \Re e\, \sigma_1(n-1) < \cdots < \Re e\, \sigma_1(1) < 0. \end{aligned} \tag{20}$$

Theorem

We suppose that condition (16) holds. The semigroup $\{T(t)\}_{t\geq 0}$ decays exponentially to zero,

$$\|T(t)\| \leq Ne^{\mu t}, \qquad t \geq 0, \tag{21}$$

where N is a positive constant and μ is the optimal decay rate given by

$$\mu = \sigma_0(1) = -\frac{1}{3}\Big(\eta - \frac{C(1)}{2} - \frac{\delta_0(1)}{2C(1)}\Big) < 0. \tag{22}$$

M. Aouadi and I. Mahfoudhi and T. Moulahi,(2022) Approximate controllability of nonsimple elastic plate with memory. *Discrete and Continuous Dynamical Systems - S* vol. 15, 5 pp 1015-1043.

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Chebyshev differentiation

Definition (Chebyshev points)

We introduced the Chebyshev-Gauss-Lobatto points defined by

$$x_i = cos(\frac{i\pi}{N}), \qquad i = 0, 1, \ldots, N. \tag{23}$$

it is the solution of the polynomial

$$T_k(x) = cos(k\, arcos(x)) \quad \text{if} \quad |x| \leq 1 \tag{24}$$

The Chebyshev polynomials of the first kind are defined by the relation

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x). \tag{25}$$

Discrete derivative by Chebyshev points

Definition (Chebyshev points)

Gvien a grid function **F** defined on the Chebyshev points, we obtain a discrete derivative **DF** in two steps :

1. Let P be the unique polynomial of degree $\leq N$ with $P(x_j) = \mathbf{F_j}, \;\; 0 \leq j \leq N$.
2. Set $\mathbf{DF_j} = P'(x_j). \quad \Longleftrightarrow \quad \mathbf{DF} = D_N \times \mathbf{F}$.

$$P(x) = \sum_{i=0}^{N} L_i(x)\mathbf{F_i} \quad \text{where} \quad L_i(x_j) = \delta_{i,j} \;\; \text{(Lagrange Polynomial)} \qquad (26)$$

This operation is linear, so it can be represented by multiplication by $(N+1) \times (N+1)$ matrix, which we shall denote by D_N.

Theorem (Chebyshev differentiation matrix.) L.N. Trefethen [1]

For each $N \geq 1$, the Chebyshev spectral differentiation matrix D_N it is defined by :

$$\begin{array}{l} (D_N)_{00} = \frac{2N^2+1}{6}, \quad (D_N)_{NN} = -\frac{2N^2+1}{6}. \\ (D_N)_{jj} = \frac{-x_j}{2(1-x_j^2)}, \quad j = 1, \ldots, N-1. \\ (D_N)_{ij} = \frac{c_i}{c_j} \frac{(-1)^{i+j}}{(x_i - x_j)}, \quad i \neq j = 1, \ldots, N-1. \end{array} \tag{27}$$

where

$$c_i = \begin{cases} 2 & i = 0, \text{ or } N. \\ 1 & \text{otherwise} \end{cases} \tag{28}$$

We present an approach based on the spectral method for spatial discretization and we use the Euler decomposition for the time variable. We denote by $\mathcal{U}^n = \left(\widetilde{u}(y_i, t_n)_{0\leq i\leq N}\right)$ and $\mathcal{W}^n = \Delta\mathcal{U}^n = \left(w(y_i, t_n)_{0\leq i\leq N}\right)$ the solutions evaluated at the Chebyshev collocation points. Thus, for $n = 1, \ldots, Nt$, becomes

$$\frac{1}{dt^2}\left(\mathcal{U}^{n+1} - 2\mathcal{U}^n + \mathcal{U}^{n-1}\right) - \alpha\mathcal{W}^n + \gamma\Delta\mathcal{W}^n = F[\mathcal{W}^n], \tag{29}$$

where $\mathcal{W}^n = \Delta\mathcal{U}^n$ and the memory term $F[\mathcal{W}^n]$ can be discretized by the trapezoidal rule, as

$$\begin{aligned} F[\mathcal{W}^n] &= \beta dt\Big(\tfrac{1}{2}\left(e^{-\eta t_n}(-\mathcal{W}^0 + \Delta\mathcal{W}^0) + (-\mathcal{W}^n + \Delta\mathcal{W}^n)\right) \\ &+ \sum_{i=1}^{n-1} e^{-\eta t_{n-i}}(-\mathcal{W}^i + \Delta\mathcal{W}^i)\Big). \end{aligned} \tag{30}$$

Thus, we can write this method in the following algorithm.

Algorithm 1 based on spectral method

Initialize :

$n = 0$, we note by $\mathcal{U}^0 = \left(u(x_j, 0)_{0\leq j\leq N}\right)$ and $\mathcal{W}^0 = \widetilde{\Delta}\mathcal{U}^0$.

$n = 1$, we note by $\mathcal{U}^1 = \left((u^0(x_j) + dtu^1(x_j))_{0\leq j\leq N}\right)$ and $\mathcal{W}^1 = \widetilde{\Delta}\mathcal{U}^1$,

for $n = 1, \ldots, Nt$ **do**

Step 1. Compute the second member of the system defined in (30) by

$$F[\mathcal{W}^n] = \beta dt\Big(\frac{1}{2}\left(e^{-\eta t_n}\widetilde{L}\mathcal{W}^0 + \widetilde{L}\mathcal{W}^n\right) + \sum_{i=1}^{n-1} e^{-\eta t_{n-i}}\widetilde{L}\mathcal{W}^i\Big) \tag{31}$$

Step 2. Solve the equation (29) and we define $\mathcal{U}^{n+1}$ by the solution $\mathcal{U}$ at $t = t_{n+1}$, then inject the boundary condition

$$\begin{aligned}&\mathcal{U}^{n+1} = 2\mathcal{U}^n - \mathcal{U}^{n-1} + dt^2\Big(F[\mathcal{W}^n] + \alpha\mathcal{W}^n - \gamma\widetilde{\Delta}\mathcal{W}^n\Big).\\&\mathcal{U}^{n+1}(-1) = \mathcal{U}^{n+1}(1) = 0\end{aligned} \tag{32}$$

Step 3. Compute the solution $\mathcal{W}$ at $t = t_{n+1}$ and inject the boundary condition.

$$\mathcal{W}^{n+1} = \widetilde{\Delta}\mathcal{U}^{n+1} \text{ and } \mathcal{W}^{n+1}(-1) = \mathcal{W}^{n+1}(1) = 0. \tag{33}$$

end for

For the example 1D, we choose the values $T = 5$, $\eta = 10$, $\beta = 8$, $\alpha = 2$ and $\gamma = 1$ with the following initial data,

$$u(x,0) = \sin(\pi x) \quad \text{and} \quad u_t(x,0) = 0, \qquad x \in [0,1].$$

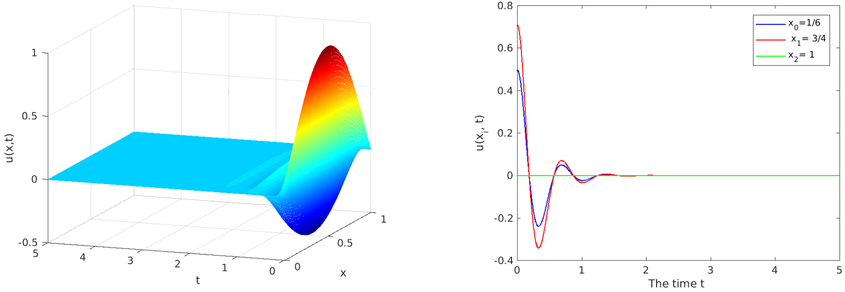
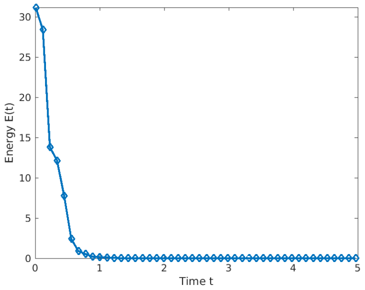


FIGURE – The displacement $u(x,t)$ for time interval $[0,5]$ (left) on the points $x_0 = 1/6$, $x_1 = 3/4$ and $x_2 = 1$ (right)



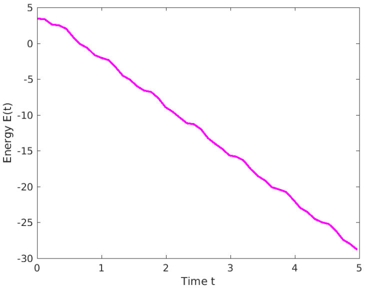


FIGURE – The Energy $E(t)$ in the time interval $[0,5]$ (left) the logarithmic energy (right)

Figure 1 shows that the displacement $u(x,t)$ rapidly decreases to zero when time increases.

In this example, we choose $T = 20, \quad \eta = 0.9, \quad \beta = 5 \times 10^{-3}$, $\alpha = 5 \times 10^{-2}$ and $\gamma = 9 \times 10^{-3}$ and the following initial data

$$u(x, y, 0) = \sin(\pi x)\sin(\pi y) \quad \text{and} \quad u_t(x, y, 0) = 0, \qquad (x, y) \in [0, 1]^2.$$

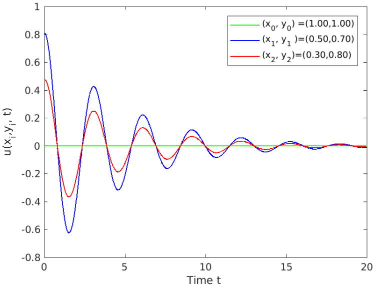
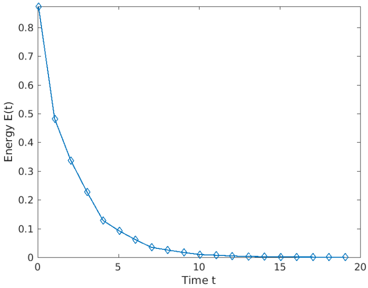


FIGURE – The displacement $u(x_i, y_i, t)$ at $(x_i, y_i) = (1, 1)$, $(\frac{1}{2}, \frac{3}{4})$ and $(\frac{1}{3}, \frac{5}{6})$, for time $[0, 20]$.



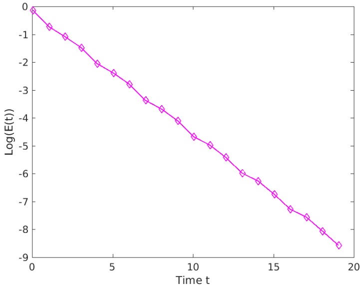


FIGURE – The energy $E(t)$ in the time interval $[0, 20]$ (left) the logarithmic energy (right)

Based, on the `roots` MATLAB function, one can plot the eigenvalues solutions to (14). We show the distribution of the eigenvalues $\sigma_0(n)$ and $\Re e(\sigma_1)(n) = \Re e(\sigma_2)(n)$ in the case $\alpha - \gamma \geq 0$. To validate the asymptotic development we choose the following values $\eta = 10, \quad \beta = 8, \quad \alpha = 2$ and $\gamma = 1$.

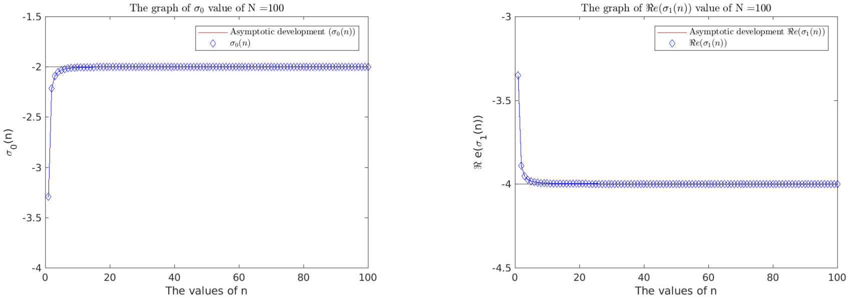


FIGURE – The eigenvalues $n \mapsto \sigma_0(n)$ and $n \mapsto \Re e(\sigma_1)(n)$

Thank you for your attention