Spectral analysis and numerical results in non-simple elastic memory plate

Imed MAHFOUDHNI

National School of Engineers of Monastir

Collaboration with Prof. Moncef Aouadi at Carthage University (Tunisia)

Control & Inverse Problems (CIP)
May 09-11, 2022 - Hotel Monatir Center, Monastir, Tunisia
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The linear anti-plane shear equations of nonsimple viscoelasticity corresponding to the isotropic homogeneous case.

\[
\begin{align*}
    u_{tt}(x, t) - \alpha \Delta u(x, t) + \gamma \Delta^2 u(x, t) \\
    + \int_0^\infty \left( m_1(s) \Delta u(x, t-s) - m_2(s) \Delta^2 u(x, t-s) \right) ds = 0
\end{align*}
\]

in \( \Omega \times [0, \infty) \), where \( \Omega \) is a bounded domain in \( \mathbb{R}^2 \) with smooth boundary \( \partial \Omega \). Where \( u \) represents the vertical displacement of the plate, \( \alpha \) and \( \gamma \) are the material coefficients and \( m_1 \) and \( m_2 \) are the memory kernels.

- **Dirichlet boundary conditions**

\[
u(x, t) = \Delta u(x, t) = 0, \quad x \in \partial \Omega, \quad t \geq 0, \quad (1)
\]

- **Initial conditions**

\[
u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x), \quad x \in \Omega, \quad (2)
\]
Hypothesis

Concerning the memory kernels $m_i$, $i = 1, 2$, we assume that

$(\mathbb{H}_1)$ \quad $m_i \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$,

$(\mathbb{H}_2)$ \quad $m_i(s) \geq 0 \quad \forall s \in \mathbb{R}^+$,

$(\mathbb{H}_3)$ \quad $m_i'(s) \leq 0 \quad \forall s \in \mathbb{R}^+$,

$(\mathbb{H}_4)$ \quad $m_i'(s) + \delta_i m_i(s) \leq 0$ for some $\delta_i > 0$, \quad $\forall s \in \mathbb{R}^+$,

$(\mathbb{H}_5)$ \quad $m_i(0) = \int_0^\infty m_i(s)ds := m_i^0 > 0$. 


Introducing the new variable $v = u_t$, setting $U = (u, v, \zeta)$, the problem can be written as a linear evolution equation in $\mathcal{H}$ of the form

$$\frac{dU}{dt} = AU, \quad U(0) = U_0,$$

(3)

where $U(0) = (u_0, u_1, \zeta_0) \in \mathcal{H}$ and $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is the linear operator defined by

$$A \begin{pmatrix} u \\ v \\ \zeta \end{pmatrix} = \begin{pmatrix} v \\ -\alpha Au - \gamma A^2 u - \int_0^\infty m_1(s) A\zeta(s)ds - \int_0^\infty m_2(s) A^2 \zeta(s) ds \\ v - \partial_s \zeta \end{pmatrix},$$

(4)

with the domain

$$D(A) = \left\{ U \in \mathcal{H} \mid \begin{array}{l} v \in V_2 \\ \alpha Au + \gamma A^2 u + \int_0^\infty m_1(s) A\zeta(s)ds + \int_0^\infty m_2(s) A^2 \zeta(s) ds \in V_0 \\ \partial_s \zeta \in \mathcal{W}, \; \zeta(0) = 0 \end{array} \right\}.$$
Lemma

Suppose that conditions \((\mathbb{H}_1) - (\mathbb{H}_5)\) hold, then the operator \(A\) generates a semigroup of contractions in \(\mathcal{H}\).

Proof: We start by showing that operator \(A\) is dissipative. Let \(U = (u, v, \zeta)\) be in \(\mathcal{D}(A)\). From (4), the divergence theorem and the boundary conditions, it is quite easy to check that

\[
< AU, U >_{\mathcal{H}} = - < \partial_s \zeta, \zeta >_{\mathcal{W}}
\]

\[
= - \frac{1}{2} \int_0^\infty m_1(\sigma) \frac{d}{d\sigma} \|\zeta(\sigma)\|_1^2 d\sigma - \frac{1}{2} \int_0^\infty m_2(\sigma) \frac{d}{d\sigma} \|\zeta(\sigma)\|_2^2 d\sigma
\]

\[
= \frac{1}{2} \int_0^\infty m'_1(\sigma) \|\zeta(\sigma)\|_1^2 d\sigma + \frac{1}{2} \int_0^\infty m'_2(\sigma) \|\zeta(\sigma)\|_2^2 d\sigma
\]

\[
\leq - \frac{\kappa_1}{2} \|\zeta\|_{\mathcal{M}_1}^2 - \frac{\kappa_2}{2} \|\zeta\|_{\mathcal{M}_2}^2 < 0.
\]

This proves that \(A\) is a dissipative operator.
Secondly, we show that the operator $A$ has the property that \( \text{Range } (I - A) = \mathcal{H} \), the operator $A$ is maximal dissipative in $\mathcal{H}$.

Since $\mathcal{D}(A)$ is densely defined in $\mathcal{H}$, from the Lumer-Phillips corollary to the Hille-Yosida theorem [1], we conclude that the operator $A$ generates a semigroup of contractions in $\mathcal{H}$.

Now an application of the theory of semigroups (see Pazy [1]) gives

**Theorem**

Suppose that conditions $(\mathbb{H}_1) - (\mathbb{H}_5)$ hold, the operator $A$ generates a $C_0$-semigroup $T(t) = e^{tA}$ on $\mathcal{H}$. Hence, the system (3) is well-posed, i.e., for any $U_0 \in \mathcal{H}$, the system (3) has a unique weak solution $U(t) = e^{tA}U_0$. Furthermore, if $U_0 \in \mathcal{D}(A)$, $U(t) = e^{tA}U_0$, becomes the classic solution to (3).

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SPECTRAL ANALYSIS OF THE PROBLEM

To simplify the analysis in the next sections, we consider that the memory kernels are given by the following form

\[ m_1(\tau) = m_2(\tau) = \beta e^{-\eta \tau}, \quad \beta, \eta > 0, \]  

(6)

Consider the positive operators \( A \) and \( A^2 \) on \( X = L^2(\Omega) \) defined by \( A\phi = -\Delta \phi \) and \( A^2\phi = \Delta^2 \phi \) with Dirichlet boundary conditions and with the domains \( \mathcal{D}(A) = H^2(\Omega) \cap H^1_0(\Omega) \) and \( \mathcal{D}(A^2) = H^4(\Omega) \). The operator \( A \) has the following very well-known properties.

(a) The spectrum of \( A = -\Delta \) consists of only eigenvalues

\[ 0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \infty, \]  

(7)

each one with multiplicity one.

(b) The eigenfunctions of \( A \) with Dirichlet boundary conditions are real analytic functions.
(c) For all $x \in \mathcal{D}(A)$ we have
\[
Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, \phi_n \rangle \phi_n = \sum_{n=1}^{\infty} \lambda_n E_n x,
\]
\[
\langle \cdot, \cdot \rangle \text{ is the inner product in } X = L^2(\Omega) \text{ and }
E_n x = \langle x, \phi_n \rangle \phi_n. \text{ So } \{E_n\} \text{ is a complete family of }
\text{orthogonal projections in } X \text{ and } x = \sum_{n=1}^{\infty} E_n x, \ x \in X.
\]
(d) The fractional powered spaces $X^r$ are given by
\[
X^r = \mathcal{D}(A^r) = \left\{ x \in X, \sum_{n=1}^{\infty} (\lambda_n)^{2r} \|E_n x\|^2 < \infty \right\}, \ r \geq 0
\]
\[
\left\{ \begin{array}{l}
\|x\|_{X^r} = \|A^r x\| = \left\{ \sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n x\|^2 \right\}^{1/2}, \ x \in X^r \\
A^r x = \sum_{n=1}^{\infty} \lambda_n^r E_n x.
\end{array} \right.
\]
The problem (1) can be written as a linear evolution equation in the Hilbert space $\mathcal{H}_1 = X^1 \times X \times X$ of the form

$$z' = Az, \quad z(0) = z^0,$$

(9)

where $z = (u, u_t, \varphi)$, and $A : \mathcal{D}(A) \subset \mathcal{H}_1 \to \mathcal{H}_1$ is given by

$$A = \begin{pmatrix} 0 & I & 0 \\ -(\alpha A + \gamma A^2) & 0 & \beta(A + A^2) \\ I & 0 & -\eta I \end{pmatrix}$$

(10)
Computing $Az$ yields

$$Az = \left( \begin{array}{c} \sum_{n=1}^{\infty} E_n z_1 \\ -\sum_{n=1}^{\infty} (\alpha \lambda_n + \gamma \lambda_n^2) E_n z_0 + \beta \sum_{n=1}^{\infty} (\lambda_n + \lambda_n^2) E_n z_2 \\ \sum_{n=1}^{\infty} E_n z_0 - \eta \sum_{n=1}^{\infty} E_n z_2 \end{array} \right)$$

$$= \sum_{n=1}^{\infty} \left( \begin{array}{ccc} 0 & 1 & 0 \\ -(\alpha \lambda_n + \gamma \lambda_n^2) & 0 & \beta (\lambda_n + \lambda_n^2) \\ 1 & 0 & -\eta \end{array} \right) \left( \begin{array}{ccc} E_n & 0 & 0 \\ 0 & E_n & 0 \\ 0 & 0 & E_n \end{array} \right) \left( \begin{array}{c} z_0 \\ z_1 \\ z_2 \end{array} \right)$$

$$= \sum_{n=1}^{\infty} A_n P_n z, \ z \in \mathcal{D}(A),$$

where $\{P_n\}_{n \geq 1}$ is a complete family of orthogonal projections in the Hilbert space $\mathcal{H}_1$.
\[ P_n = \begin{pmatrix} E_n & 0 & 0 \\ 0 & E_n & 0 \\ 0 & 0 & E_n \end{pmatrix}, \quad P_i P_j = \begin{cases} P_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \quad \sum_{n \geq 1} P_n = I, \]

and

\[ A_n = \begin{pmatrix} 0 & 1 & 0 \\ -(\alpha \lambda_n + \gamma \lambda_n^2) & 0 & \beta (\lambda_n + \lambda_n^2) \\ 1 & 0 & -\eta \end{pmatrix}, \quad n \geq 1. \]

The characteristic equation of \( A_n \) is given by

\[ \sigma^3 + \eta \sigma^2 + c_n \sigma + \eta c_n(\alpha, \gamma) - \beta c_{n,1} = 0, \]

where

\[ c_n = c_n(\alpha, \gamma) = \alpha \lambda_n + \gamma \lambda_n^2, \quad c_{n,1} = c_n(1, 1) = \lambda_n + \lambda_n^2. \]
**Proposition**: Let suppose that condition

\[ \eta > \frac{9}{8} \beta \max \left\{ \frac{1}{\gamma}, \frac{1 + \lambda_1}{\alpha + \lambda_1 \gamma} \right\} \quad (16) \]

holds and

\[ \delta_0(n) = \eta^2 - 3c_n, \quad \delta_1(n) = 2\eta^3 + 9(2\eta c_n - 3\beta c_{n,1}), \]

\[ C(n) = \sqrt[3]{\frac{1}{2} \left( \delta_1(n) + \sqrt{\delta_1^2(n) - 4\delta_0^3(n)} \right)} \quad (17) \]

where \( \sqrt{\cdot} \) and \( \sqrt[3]{\cdot} \) stand for the main branch of complex square and cubic roots. The spectrum of (14) consists of a sequence of conjugate pairs \( \{\sigma_1(n)\}_{n=1}^{\infty}, \{\sigma_2(n) = \overline{\sigma_1(n)}\}_{n=1}^{\infty} \) and a real sequence \( \{\sigma_0(n)\}_{n=1}^{\infty} \)

where

\[ \sigma_i(n) = -\frac{1}{3} \left( \eta + C(n)e^{\frac{2i\pi}{3}i} + \frac{\delta_0(n)}{C(n)}e^{-\frac{2i\pi}{3}i} \right), \quad i = 0, 1, 2, \quad n \geq 1, \quad (18) \]

where \( i \) is the imaginary unit \( (i^2 = -1) \). Moreover, we have

\[ \Re \sigma_i(n) < 0 \text{ for all } i = 0, 1, 2, \quad n \geq 1. \quad (19) \]
Lemma

We suppose that condition (16) holds. The asymptotic expressions of the eigenvalues $\sigma_i(n), i = 0, 1, 2, n \geq 1$, of (14) are given by

$$\sigma_0(n) = \frac{\beta}{\gamma} - \eta + O(\lambda_n^{-1}) \quad \text{and} \quad \sigma_1(n) = -\frac{\beta}{2\gamma} - i\left(\frac{\alpha}{2\sqrt{\gamma}} + \sqrt{\gamma}\lambda_n\right) + O(\lambda_n^{-1}),$$

as $n \to \infty$. 
Remark : We see that under the condition (16), we have \( \sigma_i(n) < 0 \) for all \( i = 0, 1, 2 \) and \( n \geq 1 \), which agrees with (19).
In the following we use Lemma 2 to show that \( \Re \sigma_i(n) \) is strictly monotone.

**Lemma**

We suppose that condition (16) holds. Then, \( \{\Re \sigma_i(n)\}_{n \geq 1} \) is strictly monotone, more precisely \( \{\sigma_0(n)\}_{n \geq 1} \) is strictly increasing and \( \{\Re \sigma_1(n)\}_{n \geq 1} = \{\Re \sigma_2(n)\}_{n \geq 1} \) is strictly decreasing with

\[
\begin{align*}
\sigma_0(1) &< \cdots < \sigma_0(n - 1) < \sigma_0(n) \\
\sigma_0(n) &< \Re \sigma_1(n) < \Re \sigma_1(n - 1) < \cdots < \Re \sigma_1(1) < 0.
\end{align*}
\]
We suppose that condition (16) holds. The semigroup \( \{T(t)\}_{t \geq 0} \) decays exponentially to zero,

\[
\|T(t)\| \leq Ne^{\mu t}, \quad t \geq 0,
\]

where \( N \) is a positive constant and \( \mu \) is the optimal decay rate given by

\[
\mu = \sigma_0(1) = -\frac{1}{3} \left( \eta - \frac{C(1)}{2} - \frac{\delta_0(1)}{2C(1)} \right) < 0.
\]

**Definition (Chebyshev points)**

We introduced the Chebyshev-Gauss-Lobatto points defined by

$$x_i = \cos\left(\frac{i\pi}{N}\right), \quad i = 0, 1, \ldots, N.$$  \hspace{1cm} (23)

it is the solution of the polynomial

$$T_k(x) = \cos(k \arccos(x)) \quad \text{if} \quad |x| \leq 1$$  \hspace{1cm} (24)

The Chebyshev polynomials of the first kind are defined by the relation

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x).$$  \hspace{1cm} (25)
Definition (Chebyshev points)

Given a grid function $F$ defined on the Chebyshev points, we obtain a discrete derivative $DF$ in two steps:

1. Let $P$ be the unique polynomial of degree $\leq N$ with $P(x_j) = F_j$, $0 \leq j \leq N$.

2. Set $DF_j = P'(x_j)$. $\iff DF = D_N \times F$.

$$P(x) = \sum_{i=0}^{N} L_i(x)F_i \quad \text{where} \quad L_i(x_j) = \delta_{i,j} \quad \text{(Lagrange Polynomial)} \quad (26)$$

This operation is linear, so it can be represented by multiplication by $(N + 1) \times (N + 1)$ matrix, which we shall denote by $D_N$. 

Theorem (Chebyshev differentiation matrix.) L.N. Trefethen [1]

For each $N \geq 1$, the Chebyshev spectral differentiation matrix $D_N$ is defined by:

\[
(D_N)_{00} = \frac{2N^2+1}{6}, \quad (D_N)_{NN} = -\frac{2N^2+1}{6}.
\]

\[
(D_N)_{jj} = \frac{2}{(1-x_j^2)}, \quad j = 1, \ldots, N - 1.
\]

\[
(D_N)_{ij} = \frac{c_i}{c_j} \frac{(-1)^{i+j}}{x_i-x_j}, \quad i \neq j = 1, \ldots, N - 1.
\]

where

\[
c_i = \begin{cases} 
2 & \text{i = 0, or } N. \\
1 & \text{otherwise}
\end{cases}
\]
We present an approach based on the spectral method for spatial discretization and we use the Euler decomposition for the time variable. We denote by $\mathcal{U}^n = (\tilde{u}(y_i, t_n)_{0 \leq i \leq N})$ and $\mathcal{W}^n = \Delta \mathcal{U}^n = (w(y_i, t_n)_{0 \leq i \leq N})$ the solutions evaluated at the Chebyshev collocation points. Thus, for $n = 1, \ldots, Nt$, becomes

$$\frac{1}{dt^2} (\mathcal{U}^{n+1} - 2\mathcal{U}^n + \mathcal{U}^{n-1}) - \alpha \mathcal{W}^n + \gamma \Delta \mathcal{W}^n = F[\mathcal{W}^n], \quad (29)$$

where $\mathcal{W}^n = \Delta \mathcal{U}^n$ and the memory term $F[\mathcal{W}^n]$ can be discretized by the trapezoidal rule, as

$$F[\mathcal{W}^n] = \beta dt \left( \frac{1}{2} (e^{-\eta t_n} (-\mathcal{W}^0 + \Delta \mathcal{W}^0) + (-\mathcal{W}^n + \Delta \mathcal{W}^n)) \right. + \left. \sum_{i=1}^{n-1} e^{-\eta t_{n-i}} (-\mathcal{W}^i + \Delta \mathcal{W}^i) \right). \quad (30)$$
Thus, we can write this method in the following algorithm.

**Algorithm 1 based on spectral method**

Initialize:

\[ n = 0, \text{ we note by } U^0 = (u(x_j, 0)_{0 \leq j \leq N}) \text{ and } \mathcal{W}^0 = \tilde{\Delta} U^0. \]

\[ n = 1, \text{ we note by } U^1 = ((u^0(x_j) + dtu^1(x_j))_{0 \leq j \leq N}) \text{ and } \mathcal{W}^1 = \tilde{\Delta} U^1, \]

for \( n = 1, \ldots, Nt \) do

Step 1. Compute the second member of the system defined in (30) by

\[
F[\mathcal{W}^n] = \beta dt \left( \frac{1}{2} (e^{-\eta^n}LW^0 + LW^n) + \sum_{i=1}^{n-1} e^{-\eta^{n-i}}LW^i \right) \tag{31}
\]

Step 2. Solve the equation (29) and we define \( U^{n+1} \) by the solution \( U \) at \( t = t_{n+1} \), then inject the boundary condition

\[
U^{n+1} = 2U^n - U^{n-1} + dt^2 \left( F[\mathcal{W}^n] + \alpha \mathcal{W}^n - \gamma \tilde{\Delta} \mathcal{W}^n \right).
\]

\[
U^{n+1}(-1) = U^{n+1}(1) = 0 \tag{32}
\]

Step 3. Compute the solution \( \mathcal{W} \) at \( t = t_{n+1} \) and inject the boundary condition.

\[
\mathcal{W}^{n+1} = \tilde{\Delta} U^{n+1} \text{ and } \mathcal{W}^{n+1}(-1) = \mathcal{W}^{n+1}(1) = 0. \tag{33}
\]

end for
For the example 1D, we choose the values $T = 5$, $\eta = 10$, $\beta = 8$, $\alpha = 2$ and $\gamma = 1$ with the following initial data,

$$u(x, 0) = \sin(\pi x) \quad \text{and} \quad u_t(x, 0) = 0, \quad x \in [0, 1].$$

**Figure** – The displacement $u(x, t)$ for time interval $[0, 5]$ (left) on the points $x_0 = 1/6$, $x_1 = 3/4$ and $x_2 = 1$ (right).
**Figure** – The Energy $E(t)$ in the time interval $[0, 5]$ (left) the logarithmic energy (right)

Figure 1 shows that the displacement $u(x, t)$ rapidly decreases to zero when time increases.
In this example, we choose $T = 20$, $\eta = 0.9$, $\beta = 5 \times 10^{-3}$, $\alpha = 5 \times 10^{-2}$ and $\gamma = 9 \times 10^{-3}$ and the following initial data

$$u(x, y, 0) = \sin(\pi x) \sin(\pi y) \quad \text{and} \quad u_t(x, y, 0) = 0, \quad (x, y) \in [0, 1]^2.$$
**Figure** – The energy $E(t)$ in the time interval $[0, 20]$ (left) the logarithmic energy (right)
Based, on the roots MATLAB function, one can plot the eigenvalues solutions to (14). We show the distribution of the eigenvalues $\sigma_0(n)$ and $\Re(\sigma_1)(n) = \Re(\sigma_2)(n)$ in the case $\alpha - \gamma \geq 0$. To validate the asymptotic development we choose the following values $\eta = 10$, $\beta = 8$, $\alpha = 2$ and $\gamma = 1$.
Thank you for your attention