New perspectives of Special Functions in Partial Pole Placement for Infinite Dimensional Systems

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Outlines

- Prerequisites and motivations
  - Spectral properties of delay systems
  - Finite pole-placement (FPP)
  - Multiple roots coalescence
  - Kummer Hypergeometric functions
- Multiplicity-Induced-Dominancy (MID) property for TDS
- P3δ Software
- Applicative perspectives
- A breach for systematic PID tuning in infinite dimension
- The use of the MID in the control of certain PDEs

Categories of systems with Time-delay

First-order linear Differential-difference equation

$$a_0\dot{y}(t) + a_1\dot{y}(t-\tau) + b_0 y(t) + b_1 y(t-\tau) = 0, \qquad (\star)$$

An equation of the form $(\star)$ is said to be :

- of retarded type if $a_0 \neq 0$ and $a_1 = 0$,
- of neutral type if $a_0 \neq 0$ and $a_1 \neq 0$,
- of advanced type if $a_0 = 0$ and $a_1 \neq 0$.

Time-delay Systems with discrete (constant) delays

$$\dot{x}(t) + \sum_{k=1}^{N} A_k \dot{x}(t-\tau_k) = \sum_{k=0}^{N} B_k x(t-\tau_k) \tag{1}$$

- $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ the state-vector
- initial conditions belonging to the Banach space $\mathcal{C}([-\tau_N, 0], \mathbb{R}^n)$.
- τ_j, $j = 1 \ldots N$ are s.t. $\tau_0 = 0$ and $0 < \tau_1 < \tau_2 < \ldots < \tau_N$
- A_j, $B_j \in \mathcal{M}_n(\mathbb{R})$ for $j = 0 \ldots N$.

Spectral properties and obstacles

- Characteristic function of (1) : $\Delta : \ \mathbb{C} \times \mathbb{R}^N{}_+ \to \mathbb{C}$
- $\Delta(s) = \det \left(s \left(I + \sum_{k=1}^{N} A_k \, e^{-\tau_k s} \right) - \sum_{k=0}^{N} B_k \, e^{-\tau_k s} \right) = \sum_{k=0}^{\tilde{N}} P_k(s) \, e^{\sigma_k s}$
- Spectrum $\chi = \chi_+ \cup \chi_0 \cup \chi_-$ zeros of Δ
- Zero solution of (1) is AS if $\chi = \chi_-$

Retarded equations spectrum distribution

Consider

$$\dot{x} = A_0 x(t) + A_1 x(t-\tau), \tag{2}$$

$$\Delta(s) = Q_0(s) + Q_\tau(s)\, e^{-\tau s},$$

The degree of a quasipolynomial is the sum of the involved polynomials plus the number of delays

Proposition

If s is a characteristic root of system (2) and deg$(Q_0) >$ deg(Q_τ), then it satisfies

$$|s| \leq ||A_0 + A_1\, e^{-\tau s}||_2. \tag{3}$$

The above proposition combined with the triangular inequality provides a generic envelope curve around the characteristic roots corresponding to (2).

Retarded-type AS $\equiv$ ES

W. Michiels and S.-I. Niculescu.
Stability and stabilization of time-delay systems,
ser. Advances in Design and Control. SIAM, 2007.

Spectrum envelope

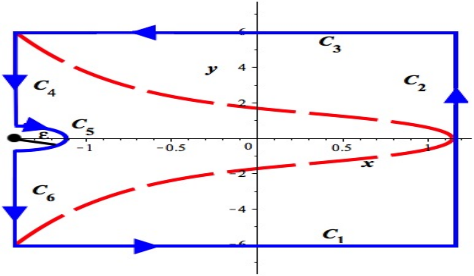


Figure: The dashed red line gives the generic spectrum envelope. In solid blue, a simplified contour for applying the argument principle to count the roots of the the quasipolynomial in the region.

Wright-Hayes Equation

The linearized system is given by :

$$\dot{x}(t) + a_0\, x(t) + a_1\, x(t-\tau) = 0,$$

with $(a_0, a_1, \tau) \in \mathbb{R}^2 \times \mathbb{R}_+^*$, the characteristic function Δ :

$$\Delta(s) = s + a_0 + a_1\, e^{-s\tau}.$$

Zero is a spectral value if and only if $a_0 + a_1 = 0$.

$$\Delta'(s,\tau) = 1 - \tau\, a_1\, e^{-s\tau},$$
$$\Delta''(s,\tau) = \tau^2\, a_1\, e^{-s\tau}.$$

The multiplicity of the zero spectral value is **at most two** reached for $\tau = 1/a_1,\ a_0 = -a_1$.
The degree of a quasipolynomial is a bound of the maximal multiplicity of its roots. Such a bound is reached only for real roots.

I. Boussaada and S-I. Niculescu.
Characterizing the codimension of zero singularities for time-delay systems.
Acta Applicandae Mathematicae, 145(1) :47–88, 2016.

Neutral equations spectrum distribution

$$\Delta(s) = Q_0(s) + Q_\tau(s)\mathrm{e}^{-s\tau} \qquad (4)$$

where $\deg(Q_0) = \deg(Q_\tau)$. Let $\alpha = \lim_{|s|\to\infty} Q_\tau(s)/Q_0(s)$:

Proposition

1. *If* $|\alpha| < 1$ *then the roots of* Δ *of large modulus are asymptotic to a vertical line* $\Re(s) \approx \log(|\alpha|)/\tau$ *in the LHP. The number of roots of* Δ *in the right of* $\Re(s) = \log(|\alpha|)/\tau + \epsilon$ *is finite* $\forall \epsilon > 0$.
2. *If* $|\alpha| > 1$ *then* Δ *has infinitely unstable roots, asymptotic to a vertical line* $\Re(s) \approx \log(|\alpha|)/\tau$ *which is in the RHP.*

J. R. Partington and C. Bonnet.
h_∞ and bibo stabilization of delay systems of neutral type.
Systems & Control Letters, 52(3) :283 – 288, 2004.

R. Bellman and K. Cooke.
Differential-difference equations.
New York : Academic Press, 1963.

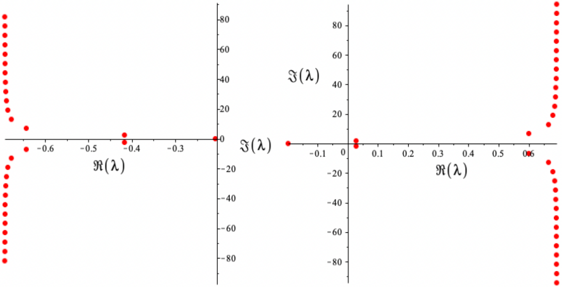


Figure: (Left) Spectrum distribution of $\Delta(s) = \left(-\frac{s^2}{2} + \frac{4s}{3} - \frac{5}{3}\right) \mathrm{e}^{-s} + s^2 + 3\,s + 3$. (Right) Spectrum distribution of $\Delta(s) = \left(-2\,s^2 + \frac{4s}{3} - \frac{5}{3}\right) \mathrm{e}^{-s} + s^2 + 3\,s + 3$.

R. Bellman and K. Cooke.
Differential-difference equations.
New York : Academic Press, 1963.

W. Michiels and S.-I. Niculescu.
Stability and stabilization of time-delay systems,
ser. Advances in Design and Control. SIAM, 2007.

Why delayed controller design ?

Control systems often operate in the presence of delays, due to the time it takes to acquire the information needed for decision-making, to create control decisions and to execute these decisions.
Consider the linear finite-dimensional system with input delay

$$\dot{x} = Ax(t) + Bu(t - \tau) \tag{5}$$

- A, B real valued matrices and τ is the delay of the system.
- A is not Hurwitz and the pair (A, B) is controllable.

Finite pole-placement method (FPP)

- Generate a prediction of the state over one delay interval :

$$x_p(t,\, t+\tau) = e^{A\tau} x(t) + \int_0^{\tau} e^{A\theta}\, B\, u(t-\theta)\, d\theta.$$

- Apply a feedback of the predicted state :

$$u(t) = K\, x_p(t,\, t+\tau).$$

- Compensating the delay effect in closed-loop

Z. Artstein.
Linear systems with delayed control : a reduction.
IEEE Transactions on Automatic Control, 27(4), 869-879, 1982.

A. Manitius and A. Olbrot.
Finite spectrum assignment problem for systems with delays.
IEEE Transactions on Automatic Control, 24 : (1), 541-552, 1979.

D Brethé, JJ Loiseau.
A result that could bear fruit for the control of delay-differential systems.
Proc. 4th IEEE Mediterranean Symp. Control Automation : 168-172, 1996.

First, rewrite

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t-\tau) \\ u(t) = Ke^{A\tau}x(t) + K\int_0^{\tau} e^{A\theta}\, B\, u(t-\theta)\, d\theta. \end{cases}$$

Using Laplace transform, one gets :

$$CE = \det \begin{bmatrix} s\,I - A & -Be^{-s\tau} \\ -Ke^{-\tau A} & I - BK\int_0^{\tau} e^{\theta(A-sI)} d\theta \end{bmatrix}$$

which gives :

$$CE = \det(s\,I - A - BK).$$

S. Mondie and W. Michiels.
Finite spectrum assignment of unstable Time-delay systems.
IEEE Transactions on Automatic Control, 48(12), 2207-2212, 2003.

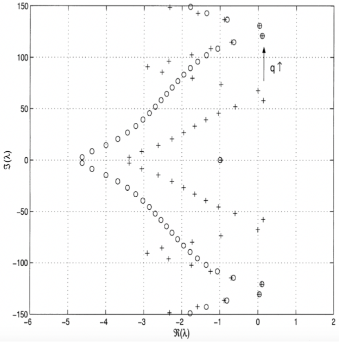
FPP limitation : An example

$$\dot{x}(t) = x(t) + u(t-1),$$

FFP suggests the controller

$$u(t) = -2\left(ex(t) + \int_0^1 e^{\theta}\, u(t-\theta)\, d\theta\right).$$

guaranteeing in closed-loop a spectral value at $s=-1$. Approximating the integral term :



$$u(t) = -2\left(e\,x(t) + \frac{1}{N}\left(\frac{u(t)}{2} + \frac{e\,u(t-1)}{2} + \sum_{l=1}^{N-1} e^{\theta}\, u(t-\frac{l}{N})\right)\right).$$

K. Engelborghs, M. Dambrine, and D. Roose.
Limitations of a class of stabilization methods for delay systems.
IEEE Transactions on Automatic Control, 46 : 336-339, Feb. 2001.

Symmetry forcing multiplicity

BAM (Bidirectional Associative Memory)

$$\begin{cases} \dot{x}_1 = -ax_1 + bx_4(t-\tau) + cx_5(t-\tau) + cx_6(t-\tau), \\ \dot{x}_2 = -ax_2 + bx_5(t-\tau) + cx_6(t-\tau) + cx_4(t-\tau), \\ \dot{x}_3 = -ax_3 + bx_6(t-\tau) + cx_4(t-\tau) + cx_5(t-\tau), \\ \dot{x}_4 = -ax_4 + bx_1(t-\tau) + cx_2(t-\tau) + cx_3(t-\tau), \\ \dot{x}_5 = -ax_5 + bx_2(t-\tau) + cx_3(t-\tau) + cx_1(t-\tau), \\ \dot{x}_6 = -ax_6 + bx_3(t-\tau) + cx_1(t-\tau) + cx_2(t-\tau), \end{cases}$$

BAM displays a dihedral group $D3$ of order 6, which is generated by the cyclic subgroup $\mathbb{Z}_3$ together with a flip of order 2.

$$\Delta(s) = \Delta_+(s).\, \Delta_-(s)$$

with $\Delta_\pm(s) = \left(s + a \pm (2c+b)e^{-s\tau}\right) . \left(s + a \pm (b-c)e^{-s\tau}\right)^2$

I. Boussaada and S. I. Niculescu.
Tracking the algebraic multiplicity of crossing imaginary roots for generic quasipolynomials : A Vandermonde-based approach.
IEEE Transactions on Automatic Control, 61 :1601-1606, 2016.

Link between $\sharp(\chi_+)$ and χ_0

$\sharp$: cardinality, $\Re$: Real part, $\Im$: Imaginary part, MU for multiplicity.
In [1], $\sharp(\chi_+)$ corresponding to a given retarded equation is established if $\chi_0 = \emptyset$. The following is proved in [2] :

Theorem

$$\sharp(\chi_+) = \frac{n - \sharp(\chi_0)}{2} + \frac{(-1)^r}{2} \text{sgn} \mathcal{I}^{(MU(0))}(0) + \sum_{j=1}^{r} \text{sgn} \mathcal{I}(\rho_j),$$

- $\mathcal{R}(y) = \Re(i^{-n}\,\Delta(i\,y))$ and $\mathcal{I}(y) = \Im(i^{-n}\Delta(iy))$
- $\rho_1, \ldots, \rho_r$ be the positive roots of $\mathcal{R}(y)$ (count multiplicity).

G. Stépán.
Retarded Dynamical Systems : Stability and Characteristic Functions.
Longman Scientific and Technical, 1989.

B.D. Hassard.
Counting roots of the characteristic equation for linear delay-differential systems.
Journal of Differential Equations, 136(2) :222 – 235, 1997.

Kummer hypergeometric functions

For $\alpha \in \mathbb{C}$ and $k \in \mathbb{N}$, $(\alpha)_k$ is the *Pochhammer symbol* for the *ascending factorial*, defined inductively as $(\alpha)_0 = 1$ and $(\alpha)_{k+1} = (\alpha + k)(\alpha)_k$.

Definition

Let $a, b \in \mathbb{C}$ and assume that b is not a nonpositive integer. The Kummer confluent hypergeometric function $\Phi(a, b, \cdot) : \mathbb{C} \to \mathbb{C}$ is the entire function defined for $z \in \mathbb{C}$ by the series

$$\Phi(a, b, z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}, \tag{6}$$

where we recall that, for $\alpha \in \mathbb{C}$ and $k \in \mathbb{N}$, $(\alpha)_k$ is the Pochhammer symbol.

The series in (6) converges for every $z \in \mathbb{C}$ and the function $\Phi(a, b, \cdot)$ satisfies the *Kummer differential equation*

$$z \frac{\partial^2 \Phi}{\partial z^2}(a, b, z) + (b - z) \frac{\partial \Phi}{\partial z}(a, b, z) - a\Phi(a, b, z) = 0 \tag{7}$$

Proposition

Let $a, b \in \mathbb{C}$ and assume that $\Re(b) > \Re(a) > 0$. Then, for every $z \in \mathbb{C}$, we have

$$\Phi(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{b-a-1} \, \mathrm{d}t, \tag{8}$$

where Γ denotes the Gamma function.

Proposition

Let $a,\ b \in \mathbb{R}$ be such that $b \geq 2$.

1. *If $b = 2a$, then all nontrivial roots z of $\Phi(a, b, \cdot)$ are purely imaginary.*
2. *If $b > 2a$, then all nontrivial roots z of $\Phi(a, b, \cdot)$ satisfy $\Re(z) > 0$.*
3. *If $b < 2a$, then all nontrivial roots z of $\Phi(a, b, \cdot)$ satisfy $\Re(z) < 0$.*
4. *If $b \neq 2a$, then all nontrivial roots z of $\Phi(a, b, \cdot)$ satisfy*

$$(b-2a)^2 \Im(z)^2 - (4a(b-a) - 2b)\, \Re(z)^2 > 0.$$

I. Boussaada, G. Mazanti and S-I. Niculescu.
Some Remarks on the Location of Non-Asymptotic Zeros of Whittaker and Kummer Hypergeometric Functions.
Bulletin des Sciences Mathématiques, 174, 2021.

Exponential decay assignment in first order equations : The MID first example

The MID property corresponds to the conditions on the system parameters under which a multiple spectral value corresponds to the spectral abscissa

$$\dot{x}(t) + a_0\, x(t) + u(t) = 0 \quad \text{where} \quad u(t) = a_1\, x(t-\tau). \tag{9}$$

Function

$$\Delta(s) = s + a_0 + a_1\, e^{-s\tau}. \tag{10}$$

admits a double root at $s = s_0$ if and only if

$$s_0 = -a_0 - \frac{1}{\tau} \text{ and } a_1 = \frac{\mathrm{e}^{-a_0\tau - 1}}{\tau}. \tag{11}$$

s_0 is the RMR. If in addition, $a_0 > -\frac{1}{\tau}$ the zero solution of system (9) is AS.

N. D. Hayes.
Roots of the transcendental equation associated with a certain difference-differential equation.
J. of the London Math. Society, s1-25(3) :226–232, 1950.

I. Boussaada, H. Unal, and S-I. Niculescu.
Multiplicity and stable varieties of time-delay systems : A missing link.
In *Proceeding of MTNS*, pages 1-6, 2016.

I. Boussaada, S-I. Niculescu, A. El-Ati, R. Pérez-Ramos and K. Trabelsi.
Multiplicity-induced-dominancy in parametric second-order delay differential equations : Analysis and application in control design.
ESAIM : COCV, 26, 57, 2020.

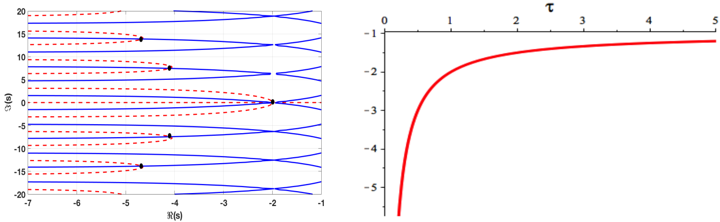


Figure: (Left)The distribution of the spectrum corresponding to $s + a_0 + \frac{\mathrm{e}^{-a_0\tau - 1}}{\tau}\,\mathrm{e}^{-s\,\tau} = 0$ for $a_0 = \tau = 1$. (Right) The rightmost root corresponding to Wright-Hayes equation as a function of the delay τ (red solid line) for a fixed value of $a_0 = 1$.

Sketch of the proof

If a_1 satisfy (11) then $\Delta(s) = s + a_0 + a_1\, e^{-s\tau}$ can be written

$$\begin{aligned}\Delta(s) &= (s - s_0)\left(1 + \frac{e^{-\tau(s-s_0)} - 1}{\tau\,(s - s_0)}\right) \\ &= (s - s_0)\left(1 - \int_0^1 e^{-\tau(s-s_0)\,t}\, dt\right)\end{aligned} \qquad (12)$$

To prove that s_0 is the spectral abscissa, let assume that there exists $s_1 = \zeta_1 + j\,\eta_1$ a root of (12) such that $\zeta_1 > s_0$. Then,

$$\begin{aligned}1 &= \int_0^1 e^{-\tau(\zeta_1 + j\,\eta_1 - s_0)\,t}\, dt = \Re\left(\int_0^1 e^{-\tau(\zeta_1 - s_0 + j\,\sigma_1)\,t}\, dt\right) \\ &\leq \left|\int_0^1 e^{-\tau(\zeta_1 - s_0 + j\,\sigma_1)\,t}\, dt\right| \leq \int_0^1 e^{-\tau(\zeta_1 - s_0)\,t}\, dt < 1,\end{aligned}$$

which proves the inconsistency of the hypothesis $\zeta_1 > s_0$.

Multiple roots are not necessarily dominant

Sparsity inducing loss of dominancy

Let revisit the problem of stabilization of a chain of integrators :

$$\Delta(s) = s^2 + \alpha\, e^{-\tau s}. \tag{13}$$

The maximal admissible multiplicity is 2 which is reached iff

$$\alpha = -4\,\frac{\mathrm{e}^{-2}}{\tau^2},\; s = -\frac{2}{\tau}.$$

$\Rightarrow$ 2 distinct delays NSC stabilize a double integrators.
$\Rightarrow$ There exists at least a root for (13) with positive real part.
$\Rightarrow$ $s_0 = -\frac{2}{\tau}$, while being a multiple root it cannot be dominant.

V.L. Kharitonov, S-I. Niculescu, J. Moreno, and W. Michiels.
Static output feedback stabilization : necessary conditions for multiple delay controllers.
IEEE Trans. on Aut. Cont., 50(1) :82–86, 2005.

S-I. Niculescu and W. Michiels.
Stabilizing a chain of integrators using multiple delays.
IEEE Trans. on Aut. Cont., 49(5) :802–807, 2004.

The generic single-delay differential equation

Consider

$$y^{(n)}(t)+a_{n-1}y^{(n-1)}(t)+\cdots+a_0y(t)+\alpha_m y^{(m)}(t-\tau)+\cdots+\alpha_0 y(t-\tau)=0, \quad (14)$$

y is real-valued, n is a positive integer, m is a nonnegative integer such that $m \leq n$, $a_k, \alpha_l \in \mathbb{R}$ for $k \in [\![0, n-1]\!]$ and $l \in [\![0, m]\!]$ are constant coefficients, and $\tau > 0$. Equation (14) is said of *retarded* type if $m < n$ and of *neutral* type if $m = n$. The asymptotic behavior of solutions (14) is characterized by the function $\Delta : \mathbb{C} \to \mathbb{C}$ defined for $s \in \mathbb{C}$ by

$$\Delta(s) = s^n + \sum_{k=0}^{n-1} a_k s^k + e^{-s\tau} \sum_{k=0}^{m} \alpha_k s^k. \quad (15)$$

Lemma

Let $s_0 \in \mathbb{R}$, Δ be the quasipolynomial from (15), and consider the quasipolynomial $\widetilde{\Delta} : \mathbb{C} \to \mathbb{C}$ obtained from Δ by the change of variables $z = \tau(s - s_0)$ and multiplication by τ^n, i.e.,

$$\widetilde{\Delta}(z) = \tau^n \Delta\left(s_0 + \tfrac{z}{\tau}\right). \tag{16}$$

Then

$$\widetilde{\Delta}(z) = z^n + \sum_{k=0}^{n-1} b_k z^k + e^{-z} \sum_{k=0}^{m} \beta_k z^k, \tag{17}$$

where

$$\begin{cases} b_k = \binom{n}{k} \tau^{n-k} s_0^{n-k} + \tau^{n-k} \sum_{j=k}^{n-1} \binom{j}{k} s_0^{j-k} a_j, & \text{for every } k \in [\![0, n-1]\!], \\ \beta_k = \tau^{n-k} e^{-s_0 \tau} \sum_{j=k}^{m} \binom{j}{k} s_0^{j-k} \alpha_j, & \text{for every } k \in [\![0, m]\!]. \end{cases} \tag{18}$$

GMID property

Theorem

Consider the quasipolynomial Δ *given by* (15) *and let* $s_0 \in \mathbb{R}$*. The number* s_0 *is a root of multiplicity* $\mathcal{D}_{PS} = m + n + 1$ *of* Δ *if and only if*

$$\begin{cases} a_k = \binom{n}{k} (-s_0)^{n-k} + (-1)^{n-k}\, n! \sum_{j=k}^{n-1} \frac{\binom{j}{k}\binom{m+n-j}{m} s_0^{j-k}}{j!\,\tau^{n-j}} & \text{for every } k \in [\![0, n-1]\!], \\ \alpha_k = (-1)^{n-1} e^{s_0 \tau} \sum_{j=k}^{m} \frac{(-1)^{j-k} (m+n-j)!\, s_0^{j-k}}{k!\,(j-k)!\,(m-j)!\,\tau^{n-j}} & \text{for every } k \in [\![0, m]\!]. \end{cases} \tag{19}$$

By considering the first equation in (19) with $k = n - 1$, one obtains the simple and interesting relation between s_0, τ, and a_{n-1} given by

$$s_0 = -\frac{a_{n-1}}{n} - \frac{m+1}{\tau}. \tag{20}$$

GMID property : Forcing the multiplicity

Lemma

Let $n \in \mathbb{N}^*$ and $m \in \mathbb{N}$ satisfy $m \leq n$, $b_0, \dots, b_{n-1}, \beta_0, \dots, \beta_m \in \mathbb{R}$, and $\widetilde{\Delta}$ be the quasipolynomial given by (17). Then 0 is a root of multiplicity $\mathcal{D}_{PS} = m + n + 1$ of $\widetilde{\Delta}$ if and only if

$$\begin{cases} b_k = (-1)^{n-k} \dfrac{n!}{k!} \dbinom{m+n-k}{m} & \text{for every } k \in [\![0, n-1]\!], \\ \beta_k = (-1)^{n-1} \dfrac{(m+n-k)!}{k!(m-k)!} & \text{for every } k \in [\![0, m]\!]. \end{cases} \tag{21}$$

Moreover, if (21) is satisfied, then, for every $z \in \mathbb{C}$,

$$\widetilde{\Delta}(z) = \frac{z^{m+n+1}}{m!} \int_0^1 t^m (1-t)^n e^{-zt} \, \mathrm{d}t. \tag{22}$$

GMID property : Dominancy

Theorem

Consider the quasipolynomial Δ *given by* (15) *and let* $s_0 \in \mathbb{R}$.

1. *(Retarded) If* $m < n$ *and* (19) *is satisfied, then* s_0 *is a strictly dominant root of* Δ.
2. *(Neutral) If* $m = n$ *and* (19) *is satisfied then,* s_0 *is a dominant root of* Δ *and, for every other complex root* s *of* Δ*, one has* $\Re(s) = s_0$. *More precisely, the set of roots of* Δ *is* $\{s_0 + i\frac{\zeta}{\tau}$ *such that* $\zeta \in \Xi_n\}$, *where*

$$\Xi_n = \left\{ \zeta \in \mathbb{R} \text{ such that } \tan\left(\frac{\zeta}{2}\right) = \frac{\zeta \displaystyle\sum_{\ell=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{\ell} \frac{(2n-2\ell-1)!}{(2\ell+1)!(n-2\ell-1)!} \zeta^{2\ell}}{\displaystyle\sum_{\ell=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{\ell} \frac{(2n-2\ell)!}{(2\ell)!(n-2\ell)!} \zeta^{2\ell}} \right\}.$$

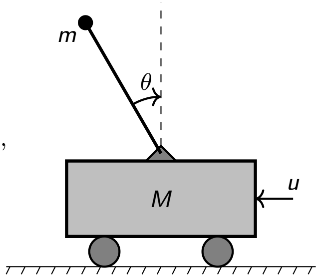
Stabilizing delayed action

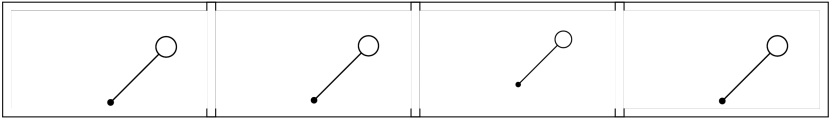
Delayed PD controller : $u(t) = k_p\, \theta(t-\tau) + k_d\, \dot{\theta}(t-\tau)$

The adimensional dynamics of the inverted pendulum is governed

$$\left(1 - \frac{3\epsilon}{4}\cos^2(\theta)\right)\ddot{\theta} + \frac{3\epsilon}{8}\dot{\theta}^2 \sin(2\theta) - \sin\theta + u\cos\theta = 0,$$

où $\epsilon = m/(m+M)$.





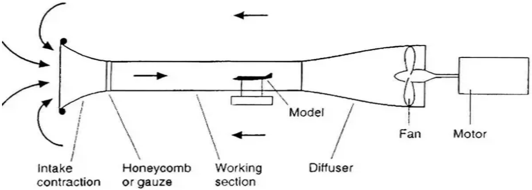
$\tau = 0$ | $\tau = 0.13$s | $\tau = 0.17$s | $\tau = 0.17$s+MID

P3δ Software

https://cutt.ly/p3delta

Mach number regulation in a wind tunnel model

The Mach number regulation in a wind tunnel is based on the Navier-Stokes equations for unsteady flow and contains control laws for temperature and pressure regulation.



The following simplified model of Mach number regulation described in [1] consists of :

$$\begin{cases} \dot{\xi}_1(t) = -a\xi_1(t) + k\,a\,\xi_2(t-\tau) \\ \dot{\xi}_2(t) = \xi_3(t) \\ \dot{\xi}_3(t) = -\omega^2\,\xi_2(t) - 2\zeta\omega\xi_3(t) + \omega^2 u(t) \end{cases} \tag{23}$$

ξ_1 the dynamic response of the Mach number perturbations, ξ_2 a small perturbations in the guide vane angle actuator a, ω, ζ, k and τ are positive parameters depending on the operating point and presumed constant when the perturbations ξ_i are small.

A. Manitius.
Feedback controllers for a wind tunnel model involving a delay : Analytical design and numerical simulation.
IEEE TAC, 29(12) :1058–1068, 1984.

Consider the control law : $u(t) = -\frac{\alpha}{\omega^2}\xi_2(t) - \frac{\beta_0}{\omega^2}\xi_2(t-\tau) - \frac{\beta_1}{\omega^2}\xi_3(t-\tau)$. In our case, the corresponding quasipolynomial function is given by :

$$\Delta(s) = (s+a)\left((s\beta_1+\beta_0)\mathrm{e}^{-s\tau} + s^2 + 2\,s\,\zeta\,\omega + \omega^2 + \alpha\right). \tag{24}$$

$$s_- = \frac{-2 - \zeta\,\omega\,\tau}{\tau}$$

is the rightmost root of the second factor of function (24), which insures the stability of the steady state solution.

I. Boussaada, S-I. Niculescu, and K. Trabelsi.
Toward a decay rate assignment based design for time-delay systems with multiple spectral values.
In *Proceeding of MTNS*, pages 864–871, 2018.

Piezo-actuated flexible beam, clamped at one edge

- Euler-Bernoulli PDE modelling +Neumann and Dirichlet BC (coupled PDEs + nonlinear BC).
- Finite Element Modelling (huge number of dof).

$$
\begin{aligned}
\mathbb{M}_{qq}\ddot{q}(t) + \mathbb{D}_{qq}\dot{q}(t) + \mathbb{K}_{qq}q(t) &= \mathbb{M}_{qw}w(t) - \mathbb{K}_{qu}u(t) \\
y(t) &= \mathbb{K}_{qy}q(t) \\
z(t) &= \mathbb{F}_{zw}w(t) - \mathbb{F}_{zu}u(t) - \mathbb{F}_{zq}q(t) - \mathbb{F}_{zv}\dot{q}(t)
\end{aligned}
$$

- Modal Analysis

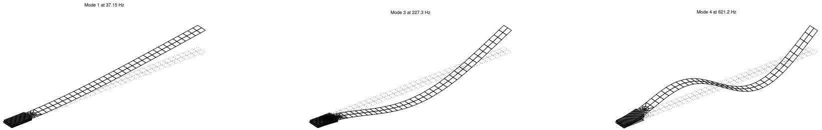


Figure: Three controllable & observable modes

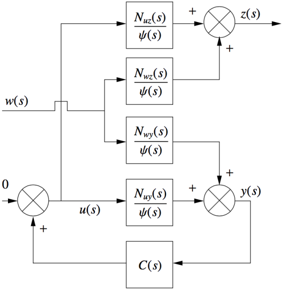
In Laplace domain, the transfert functions :

$$z(s) = \frac{N_{wz}(s)}{\psi(s)} w(s) + \frac{N_{uz}(s)}{\psi(s)} u(s)$$

$$y(s) = \frac{N_{wy}(s)}{\psi(s)} w(s) + \frac{N_{uy}(s)}{\psi(s)} u(s)$$

$$n_0 \simeq 0.013,\ n_{r_0} \simeq 77.287$$
$$d_0 \simeq 1.001,\ d_{r_0} \simeq 6.373$$
$$\tau \simeq 0.004.$$

$s_0 \simeq -244.1$ approximate the rightmost root of multiplicity 3.



$$C(s) = \frac{N(s,\tau)}{D(s,\tau)}$$

$$N(s,\tau) = n_0 + n_{r_0}\, e^{-\tau s}$$

$$D(s,\tau) = d_0 + d_{r_0}\, e^{-\tau s}.$$

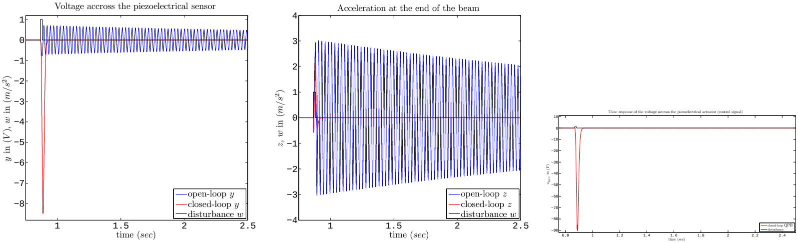


Figure: Time responses of the measured output y on the left, of the controlled output z on the middle and of the closed-loop control signal u on the right.

I. Boussaada, S-I. Niculescu, S. Tliba, and T. Vyhlídal.
On the coalescence of spectral values and its effect on the stability of time-delay systems : Application to active vibration control.
Procedia IUTAM, 22(Supplement C) :75 – 82, 2017.

The MID property : A breach for systematic PID tuning

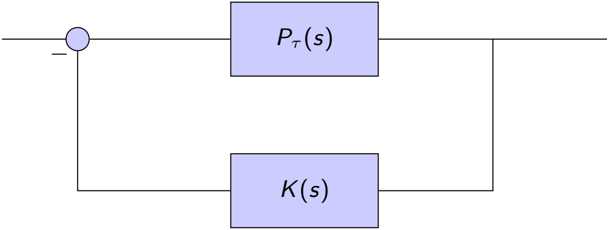


Figure: A feedback control system

where

$$P_\tau(s) = P_0(s)\, e^{-\tau\, s} \quad \text{and} \quad K_{PID}(s) = k_p + k_d\, s + \frac{k_i}{s}, \qquad (25)$$

with $P_0(s)$ is a delay-free plant and the aim is to tune the standard PID gains $(k_i,\ k_p,\ k_d)$ achieving the stabilization in closed-loop.

First order delayed plant

The resulting closed-loop plant is given by :

$$M(s) = \frac{\left(k_p\, s + k_i + s^2 k_d\right) \mathrm{e}^{-s\tau}}{s^2 - sp + \mathrm{e}^{-s\tau} k_p\, s + \mathrm{e}^{-s\tau} k_i + \mathrm{e}^{-s\tau} s^2 k_d}. \tag{26}$$

The corresponding characteristic equation :

$$\begin{cases} \Delta(s) = Q_0(s) + Q_\tau(s)\mathrm{e}^{-s\tau} \quad \text{where} \\ Q_0(s) = s^2 - sp \quad \text{and} \quad Q_\tau(s) = k_d s^2 + k_p\, s + k_i. \end{cases} \tag{27}$$

Theorem

i) For arbitrary k_p, k_i, k_d, τ, the root's multiplicity of (27) is bounded by 4.

ii) The quasipolynomial (27) admits a multiple real spectral value at

$$s_\pm = \frac{-\tau\, p - 6 \pm \sqrt{\tau^2 p^2 + 12}}{2\tau} \tag{28}$$

with algebraic multiplicity 4 if, and only if,

$$\begin{cases} k_d = \dfrac{(4 + 2\,\tau\, s_\pm - \tau\, p)\,\mathrm{e}^{\tau\, s_\pm}}{2}, \\ k_p = -\dfrac{\left(\left(8\,\tau + \tau^2 s_\pm\right) p - 18 - 12\,\tau\, s_\pm\right)\mathrm{e}^{\tau\, s_\pm}}{\tau}, \\ k_i = \dfrac{\left((\tau s_\pm + 3)\,\tau^2 p^2 + (-12\tau s_\pm - 60)\,\tau p + 108 + 84\,\tau\, s_\pm\right)\mathrm{e}^{\tau\, s_\pm}}{2\tau^2}. \end{cases} \tag{29}$$

iii) If (29) is satisfied then $s = s_+$ is the rightmost root of quasipolynomial function (27).

D. Ma, I. Boussaada, J. Chen, C. Bonnet, S-I. Niculescu and J. Chen.
PID control design for first-order delay systems via MID pole placement : Performance vs. robustness.
Automatica, 137, 2022.

Perspectives in the control of PDEs

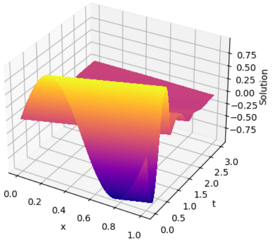
Transport and propagation

$$\partial_t \varphi(t,x) + \lambda\, \partial_x \varphi(t,x) = 0 \ \text{ with } \ (t,x) \in [0,\ \infty) \times (0,\ L).$$

λ constant and boundary conditions as a PI controller :$\varphi(t,\,0) = k_p \varphi(t,L) + k_i \int_0^t \varphi(\nu, L)\,\mathrm{d}\nu$.

In frequency domain $\Delta(s) = s - (k_i + k_p\, s) e^{-\frac{L}{\lambda}\, s}$.

MID : Forcing a triple spectrale value allows to assign the decay rate $s_0 = -2\,\lambda/L$ via le choix $k_p = -e^{-2}$ et $k_i = -4\, e^{-2}\, \lambda/L$. La condition initiale : $\varphi(0,x) = \sin(2\pi x)$ avec $\frac{L}{\lambda} = 1$.



I. Boussaada, G. Mazanti and S-I. Niculescu.
The generic multiplicity-induced-dominancy property from retarded to neutral delay-differential equations : When delay-systems characteristics meet the zeros of Kummer functions.
Comptes rendus Mathématique, 360, 349-369, 2022.

Conclusion and potential extensions

- We proposed a new pole placement paradigm based on two properties MID and CRRID.
- The ensuing control strategy is robust w.r.t uncertain parameters and easy to implement.
- Systematic perspective in PID tuning for infinite dimensional systems.
- The effectiveness of the strategy is demonstrated on mechanical engineering applications : vibration damping
- Robustness : the mechanism coalescence/splitting for TDS is described in

  W. Michiels, I. Boussaada and S-I. Niculescu.
  An explicit formula for the splitting of multiple eigenvalues for nonlinear eigenvalue problems and connections with the linearization for the delay eigenvalue problem.
  SIAM J. Matrix Analysis Applications, 38(2) :599–620, 2017.

- The effect of coexistence of distinct real roots on the AS of TDS's trivial solution (allowing to the CRRID) in

  F. Bedouhene, I. Boussaada and S-I. Niculescu.
  Real spectral values coexistence and their effect on the stability of time-delay systems : Vandermonde matrices and exponential decay.
  Comptes Rendus. Mathématique, Tome 358 (2020).

- Mechanical engineering applications : vibration damping
- Extensions to wave and transport equations opened new perspectives for decay assignment for PDEs.

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Thank you for your attention !