Dirac System, Sturm–Liouville equations and damped string

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Joint work with **Alexander Gomilko**

Damped string vibrations

If $v = v(x,t)$ is the vertical position of the string in time $t \in [0,\infty)$ on the interval $[0,1]$, then small vibrations are described by the wave equation

$$v_{tt}(x,t) - \rho(x)^{-1}\big(p(x)v_x(x,t)\big)_x + 2d(x)v_t(x,t) + q(x)v(x,t) = 0.$$

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+ initial position v_0 and velocity v_1

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This problem can be transformed into an abstract Cauchy problem on a Hilbert space.

Damped string - operator form

We take $V = [v, v_t]^T$ and then

$$V'(t) = BV(t), \quad t > 0,$$
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+ appropriate domain $\mathcal{D}(B)$.

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where

$$F(x, \mu) = \rho(x)\Big[(g(x) + 2d(x)f(x) + \mu f(x)\Big]$$
$$v(x) = f(x) + \mu u(x).$$

Damped string - SL equation

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Conclusion: asymptotical behavior of solutions of Sturm–Liouville problem is important.

Damped string - examples of results

$$v_{tt}(x,t) - \rho(x)^{-1}\big(p(x)v_x(x,t)\big)_x + 2d(x)v_t(x,t) + q(x)v(x,t) = 0.$$

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- [Rz.2013] - exponential stability of C_0-semigroup.
- [Rz.2017] - basis with parentheses, Riesz basis property + additional assumptions.
- [L.2022] - Riesz basis via Dirac system, L_1-type assumptions.
- [Rz.S.2018] - coupled string with constant coefficients, polynomial stability.
- [BA.B.2020] - coupled string with variable coefficient, Riesz basis property + further works on networks...

Sturm–Liouville equation with singular potential

$$y'' + q(x)y + \lambda y = 0, \quad x \in [0, 1],$$

where a potential $q \in W_2^{-1}[0, 1]$,

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where a potential $q \in W_2^{-1}[0,1]$,i.e.

$$q(x) = \sigma'(x), \qquad \sigma \in L_2[0,1],$$

and the derivative is understood in the sense of distributions.

Sturm–Liouville equation recasted

Following the regularization method we formally introduce a quasi-derivative $y^{[1]}$ of y:

$$y^{[1]}(x) := y'(x) + \sigma(x)y(x),$$

so that

$$y'' + q(x)y + \lambda y = 0, \quad x \in [0,1],$$

can be rewritten in the form

$$(y^{[1]})'(x) - \sigma(x)y^{[1]}(x) + \sigma^2(x)y(x) + \lambda y(x) = 0, \qquad x \in [0,1].$$

Sturm–Liouville problem $\to$ Dirac system

If we take $\lambda = \mu^2$ and $\mu \neq 0$ then the transformation

$$\begin{pmatrix} y \\ y^{[1]} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i\mu & -i\mu \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

leads to perturbed Dirac system for $V = (v_1, v_2)^T$, $x \in [0, 1]$:

$$\begin{aligned} V'(x) &+ \begin{bmatrix} 0 & \sigma(x) \\ \sigma(x) & 0 \end{bmatrix} V(x) \\ &= i\mu \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} V(x) + \frac{i\sigma^2(x)}{2} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} V(x). \end{aligned}$$

Sturm–Liouville problem → Dirac system...

In order to analyze perturbed system

$$V'(x) + \begin{bmatrix} 0 & \sigma(x) \\ \sigma(x) & 0 \end{bmatrix} V(x) = i\mu \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} V(x) + \frac{i\sigma^2(x)}{2} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} V(x)$$

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we study a general matrix Cauchy problem for the Dirac system

$$D'(x) + \begin{bmatrix} 0 & \sigma_1(x) \\ \sigma_2(x) & 0 \end{bmatrix} D(x) = i\mu \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} D(x),$$

$$D(0) = I,$$

where $\mu \in \mathbb{C}$ is a spectral parameter, $|\operatorname{Im}\mu| \leq d$, $|\mu| \to \infty$ and $\sigma_j \in L_2[0,1]$, $j = 1, 2$.

Dirac system

$$D'(x) - A_\mu D(x) = J(x) D(x), \quad D(0) = I,$$

$$A_\mu = \begin{bmatrix} i\mu & 0 \\ 0 & -i\mu \end{bmatrix}, \quad J(x) = \begin{bmatrix} 0 & \sigma_1(x) \\ \sigma_2(x) & 0 \end{bmatrix}.$$

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- [L.M.2014] - Riesz basis property, *short* formulas for $\mu \in \mathbb{C}$, $|\operatorname{Im} \mu| \leq d$, $|\mu| \to \infty$.
- [S.S.2014] - *short* formulas for $\mu \in \mathbb{C}$, $|\operatorname{Im} \mu| \leq d$, $|\mu| \to \infty$.
- [G.Rz.2020] - results on asymptotic behavior of solutions for $\mu \in \mathbb{C}$, $|\operatorname{Im} \mu| \leq d$, $|\mu| \to \infty$ $\sigma_j \in L_2[0,1]$, $j = 1,2$ + perturbed system and applications to SL problem.
- [Rz.2021] - results on asymptotic behavior of solutions for $\mu \in \mathbb{C}$, $|\operatorname{Im} \mu| \leq d$, $|\mu| \to \infty$ $\sigma_j \in L_p[0,1]$, $1 \leq p < 2$ $j = 1,2$ and applications to SL problem.

Sturm–Liouville problem

$$y''(x) + q(x)y(x) + \lambda y(x) = 0, \qquad x \in [0,1],$$
$$y(0) = 0, \quad y(1) = 0.$$

where $q = \sigma'$ and $\sigma \in L_2[0,1]$.

Theorem 3.1

If $(\lambda_n)_{n\geq 1}$ are the eigenvalues of the spectral problem, then

$$\lambda_n = \mu_n^2, \qquad \mu_n = \pi n + \mu_{0,n} + r_n, \qquad n \in \mathbb{N},$$

where

$$\begin{aligned} \mu_{0,n} := &\int_0^1 \sin(2\pi n t)\sigma(t)dt \\ &- 2\int_0^1 \int_0^t \sigma(t)\sigma(s)\sin(2\pi n s)\cos(2\pi n t)dsdt, \\ &- \frac{1}{2\pi n}\int_0^1 (1-\cos(2\pi n t))\sigma^2(t)dt, \end{aligned}$$

and $(r_n)_{n\geq 1} \in l_1$.

Theorem 3.2

The eigenfunctions $(y_n)_{n\geq 1}$ of the spectral problem satisfy

$$\pi n y_n(x) = y_{0,n}(x) + \psi_{1,n}(x),$$

$$y_{0,n}(x) = \sin(\pi n x)\left(1 + \int_0^x \cos(2\pi n s)\sigma(s)ds - \frac{1}{2\pi n}\int_0^x \sin(2\pi n t)\sigma^2(t)dt\right)$$

$$+ \cos(\pi n x)\left(\mu_{0,n}x - \int_0^x \sin(2\pi n s)\sigma(s)ds + \frac{1}{2\pi n}\int_0^x (1-\cos(2\pi n t))\sigma^2(t)dt\right.$$

$$\left. + 2\int_0^x \int_0^t \sigma(t)\sigma(s)\sin(2\pi n s)\cos(2\pi n t)dsdt\right),$$

where

$$\sup_{x\in[0,1]} \sum_{n=1}^{\infty} |\psi_{1,n}(x)| < \infty, \quad j = 1, 2.$$

Main result

Theorem 3.3 (G. Rz.)

The eigenfunctions of the spectral problem admit the representation

$$\pi n y_n(x) = y_{0,n}(x) + y_{1,n}(x) + \tilde{\psi}_{1,n}(x),$$

$$y_{1,n}(x) = \sin(\pi n x) A_n(x) + \cos(\pi n x) B_n(x)$$

$$\sum_{n=1}^{\infty} \sup_{x\in[0,1]} |\tilde{\psi}_{1,n}(x)| < \infty.$$

Remark 3.4

Our results are also applicable to the equation

$$(a(x)y'(x))' + q_0(x)y(x) + \mu^2 c(x)y(x) = 0, \qquad x \in [0,1],$$

where $q_0 = u_0'$, $u \in L_2[0,1]$, and the coefficients a, c are such that

$$a \in W_2^1[0,1], \quad c \in W_2^1[0,1], \quad a(x) > 0, \quad c(x) > 0, \quad x \in [0,1].$$

Dirac system with integrable potential

$$D'(x) - A_\mu D(x) = J(x) D(x), \quad x \in [0,1]$$
$$D(0) = I, \quad A_\mu = \begin{bmatrix} i\mu & 0 \\ 0 & -i\mu \end{bmatrix}, \quad J(x) = \begin{bmatrix} 0 & \sigma_1(x) \\ \sigma_2(x) & 0 \end{bmatrix}.$$

$\sigma_j \in L_p[0,1]$, $1 \leq p < 2$ $j = 1, 2$

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$\sigma_j \in L_p[0,1]$, $1 \leq p < 2$ $j = 1,2$
$M(B)$ - Banach space of 2×2 matrices with entries from a Banach space B and norm

$$\|Q\|_{M(B)} := \sum_{k,j=1}^{2} \|q_{jk}\|_B, \quad Q = [q_{jk}]_{j,k=1}^2.$$

Lemma 3.5

If $\sigma_j \in L_p[0,1]$, $1 \le p < 2$ then the unique solution $D = D(x,\mu)$ can be represented as

$$D(x,\mu) = e^{xA_\mu} + \int_0^x e^{(x-2t)A_\mu}[-J(t) + Q(x,t)]dt,$$

where $Q \in M(B)$ is the unique solution of the integral equation

$$Q(x,t) = \int_0^{x-t} J(t+\xi)J(\xi)d\xi) - \int_0^{x-t} J(t+\xi)Q(t+\xi,\xi)d\xi.$$

Moreover,

$$\|Q\|_{M(B)} \le c_1, \quad \|D\|_{M(C[0,1])} \le c_2.$$

Idea of the proof

$$Q(x,t) = \int_0^{x-t} J(t+\xi)J(\xi)d\xi) - \int_0^{x-t} J(t+\xi)Q(t+\xi,\xi)d\xi.$$

Idea of the proof

$$Q(x,t) = \tilde{J}(x,t) - \int_0^{x-t} J(t+\xi)Q(t+\xi,\xi)d\xi, \quad \tilde{J} \in M(B).$$

Idea of the proof

$$Q(x,t) = \tilde{J}(x,t) + \tilde{T}Q(x,t) \quad \tilde{J} \in M(B).$$

Idea of the proof

$$(I - \tilde{T})Q = \tilde{J}.$$

In particular

$$\|\tilde{T}F\|_{M(B)} \leq c\|F\|_{M(B)}, \quad F \in M(B)$$

and $Q \in M(B)$ with

$$Q = \sum_{n=0}^{\infty} \tilde{T}^n \tilde{J}.$$

Then for

$$Q = \sum_{n=0}^{\infty} \tilde{T}^n \tilde{J}.$$

$$D(x,\mu) = e^{xA_\mu} + \int_0^x e^{(x-2t)A_\mu}[-J(t) + Q(x,t)]dt,$$

Then for

$$Q = \sum_{n=0}^{\infty} \tilde{T}^n \tilde{J}.$$

$$D(x,\mu) = e^{xA_\mu} + \int_0^x e^{(x-2t)A_\mu}[-J(t) + Q(x,t)]dt,$$

we obtain the following representation:

$$\begin{aligned} D(x,\mu) = e^{xA_\mu} &- \int_0^x e^{(x-2t)A_\mu} J(t)dt \\ &+ \int_0^x e^{(x-2t)A_\mu}(\tilde{J} + \tilde{T}\tilde{J})(x,t)dt \\ &+ \int_0^x e^{(x-2t)A_\mu}\sum_{n=2}^{\infty}(\tilde{T}^n\tilde{J})(x,t)dt. \end{aligned}$$

Main result

Theorem 3.6

For every $d > 0$ there exist a constant $C_j = C_j(d, \sigma_1, \sigma_2)$, $j = 0, 1, 2$ such that for all $x \in [0, 1]$ and $\mu \in P_d$, the solutions of DS admit the following representation

$$D(x, \mu) = e^{xA_\mu} + D_0(x, \mu) + R_0(x, \mu),$$

where

$$D_0(x, \mu) := \int_0^x e^{(x-2t)A_\mu}(-J(t) + \tilde{J}(x, t))\, dt$$

and

$$\|R_0(x, \mu)\|_{\mathbb{C}^{2\times 2}} \leq C_2(\gamma_q(\mu)\gamma(x, \mu) + \widetilde{\gamma}(\mu)), \quad x \in [0, 1].$$

Main result...

$$D(x,\mu) = e^{xA_\mu} + D_0(x,\mu) + R_0(x,\mu),$$

$$\|R_0(x,\mu)\|_{\mathbb{C}^{2\times 2}} \le C\Big(\gamma_q(\mu)\gamma(x,\mu) + \widetilde{\gamma}(\mu)\Big).$$

$$\gamma_q(\mu) := \sum_{j=1}^{2}\left(\Big\|\int_0^x e^{-2i\mu t}\sigma_j(t)dt\Big\|_{L_q} + \Big\|\int_0^x e^{2i\mu t}\sigma_j(t)dt\Big\|_{L_q}\right),$$

$$\gamma(x,\mu) := \sum_{j=1}^{2}\left(\Big|\int_0^x e^{-2i\mu t}\sigma_j(t)dt\Big| + \Big|\int_0^x e^{2i\mu t}\sigma_j(t)dt\Big|\right),$$

$$\widetilde{\gamma}(\mu) := \int_0^1 \sigma_0(s)\gamma^2(s,\mu)\,ds.$$

Spectral problem

We consider a spectral problem

$$Y'(x) + J(x)Y(x) = A_\mu Y(x), \quad x \in [0,1], \tag{3.1}$$

associated with studied Dirac system where $Y = [y_1, y_2]^T$ and

$$y_1(0) = y_2(0), \;\; y_1(1) = y_2(1). \tag{3.2}$$

Spectral problem - results

Theorem 3.7

Let $1 < p < 2$. The eigenvalues of the spectral problem lie in a certain strip P_d and admit the representation

$$\mu_n = \pi n + \mu_{0,n} + \rho_n, \quad n \in \mathbb{Z},$$

$$\mu_{0,n} = \frac{1}{2i}\int_0^1 e^{-2\pi int}\sigma_1(t)dt - \frac{1}{2i}\int_0^1 e^{2\pi int}\sigma_2(t)dt - i\int_0^1\int_0^t \sigma_1(t)\sigma_2(\xi)e^{-2\pi int}e^{2\pi in\xi}d\xi dt$$

$$\sum_{n\in\mathbb{Z}} |\rho_n|^{q/2} < \infty.$$

Spectral problem - results...

Let $1 \le p < 2$, then the eigenfunctions of the spectral problem (3.1)-(3.2) admit the representation

$$y_1(x,\mu_n) = e^{i\pi nx}\Big(1 + i\mu_{0,n}x - \int_0^x e^{-2\pi int}\sigma_1(t)dt + \int_0^x \int_0^s \sigma_1(s)\sigma_2(\xi)e^{-2i\mu s}e^{2i\mu\xi}d\xi ds\Big) + r_1(x,n),$$

$$y_2(x,\mu_n) = e^{-i\pi nx}\Big(1 - i\mu_{0,n}x - \int_0^x e^{2\pi int}\sigma_2(t)dt + \int_0^x \int_0^s \sigma_1(\xi)\sigma_2(s)e^{2i\mu s}e^{-2i\mu\xi}d\xi ds\Big) + r_2(x,n),$$

$$\sup_{x\in[0,1]} \sum_{n\in\mathbb{Z}} |r_j(x,n)|^{q/2} < \infty.$$

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