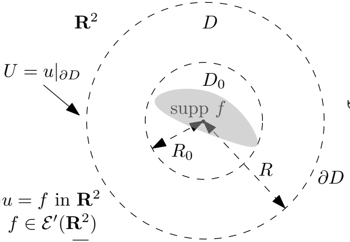
Resolution and control of solutions of multiplier equations

Workshop "Control & Inverse Problems (CIP)"
Monastir, Tunisia, 9–11 May 2022

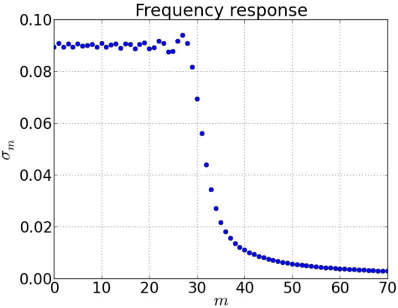
Mirza Karamehmedović

Department of Applied Mathematics and Computer Science
Technical University of Denmark
mika@dtu.dk

Motivation 1: robustness and resolution of solution of inverse source problems



$(\Delta + k^2)u = f$ in $\mathbf{R}^2$
$f \in \mathcal{E}'(\mathbf{R}^2)$
supp $f \subseteq \overline{D}_0$
+ radiation conditions



K. (2018). *J Phys Commun*

Forward operator: $U(x) = Ff(x) = \int_{y \in D_0} H_0^{(1)}(k|x-y|)f(y)$, $x \in \partial D$

Bao, Lin, & Triki (2010). *J Differ Equ*:

$$F : L^2(D_0) \xrightarrow{\text{cpct.}} L^2(\partial D), \quad F = \sum_{m \in \mathbf{Z}} \sigma_m(\cdot, \psi_m)\phi_m$$

$$\sigma_{-m} = \sigma_m, \quad \psi_m(x) \propto J_m(k|x|)e^{im\angle x}, \quad \phi_m(\angle x) \propto e^{im\angle x}$$

Bounds on the 'bandwidth' $\mathscr{B}$ of F

K. (2018). *J Phys Commun*:

Definition: $\mathscr{B} = \text{argmin}_{m\in\mathbf{N}_0}\{\sigma_{m+n} > \sigma_{m+n+1} \text{ for all } n \in \mathbf{N}_0\}$.

Theorem: $\mathscr{B} \geq \text{argmin}_{m\in\mathbf{N}_0}\{j_{m,1} \geq kR_0\}$ (tight)

Conjecture: $\mathscr{B} \leq \text{argmin}_{m\in\mathbf{N}_0}\{y_{m,1} \geq kR_0\}$ (tight)

Theorem: For the source-to-*far*-field operator, $\sigma_m = \mathcal{O}((kR_0/2)^m/m!)$ when $m \geq \text{argmin}_{m\in\mathbf{N}_0}\{y_{m,1} \geq kR_0\}$ (with explicit bound)

Kirkeby, Henriksen, & K. (2020). *Inverse Probl*:

Theorem: For the Helmholtz equation in $\mathbf{R}^3$, we have $\psi_{m,n}(x) \propto j_m(k|x|)Y_m^n(x/|x|)$ and $\phi_{m,n} \propto Y_m^n(x/|x|)$.

Theorem: $\mathscr{B} \geq \text{argmin}_{m\in\mathbf{N}_0}\{j_{m+1/2,1} \geq kR_0\}$.

Kirkeby, Henriksen, & K. (2020); K., Kirkeby, & Knudsen (2018). *Inverse Probl*:

Stability of reconstruction from a finite number of measurements in the multi-frequency ISP.

Some related work

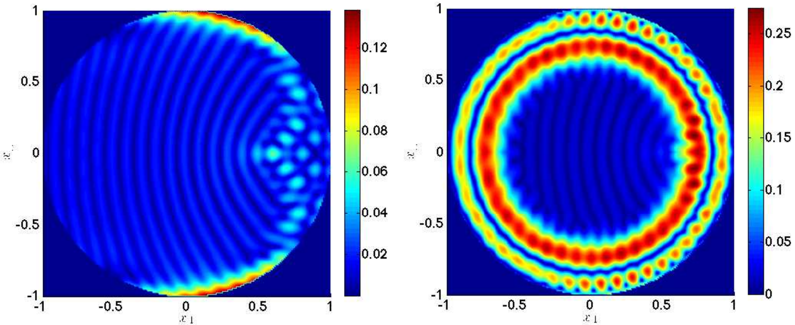
Griesmaier & Sylvester (2017). *SIAM J Appl Math*
Griesmaier & Sylvester (2016). *SIAM J Appl Math*
Griesmaier, Hanke, & Sylvester (2014). *SIAM J Numer Anal*
Griesmaier, Hanke, & Raasch (2012). *SIAM J Sci Comput*

- spectral cutoff of the source-to-far-field operator ("restricted Fourier transform") in $\mathbf{R}^2$ and $\mathbf{R}^3$; the singular values decay rapidly when $|m| \geq kR_0$.
- windowed Fourier transform
- far-field splitting and uncertainty principles for ISP

Pierri & Moretta (2020,2021). *Electronics*
Xu & Janaswamy (2006). *IEEE Trans Antennas Propag*

- spectral analysis of electromagnetic radiation operators
- applications in antenna design and measurements

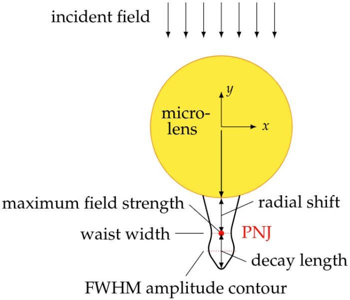
Robustness of solution of inverse source problems

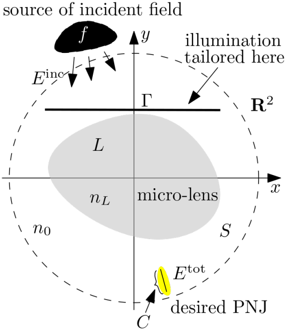


- $f^\dagger = F^\dagger U \approx \sum_{|m|\leq C} \sigma_m^{-1}(U, \phi_m)_{L^2(\partial D)}\psi_m$
- $kR = kR_0 = 10\pi$
- $\mathscr{B} \geq 26$ (K. (2018). *J Phys Commun*)
- $m_{\text{noise}} = 26$ vs. $m_{\text{noise}} = 30$, for same amplitude of noise component

Motivation 2: resolution of control of solutions of PDE

Photonic jet control by amplitude- and phase-modulated illumination of a dielectric micro-lens.





K., Scheel, Pedersen, and Hansen, *in review*.

- Define the 'lens contrast' $\alpha = k_0^2(n_L^2 - 1)$, the characteristic function

$$\chi_L(x) = \begin{cases} 1, & x \in L, \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

  and the piecewise constant wavenumber

$$k(x) = k_0[1 + \chi_L(x)(n_L - 1)], \quad x \in \boldsymbol{R}^2. \tag{2}$$

- Introduce the 'desired total field' E^{tot} as the solution of the Helmholtz problem

$$\left.\begin{array}{rcll} (\Delta + k(x)^2)E^{\text{tot}}(x) & = & 0, & x \in S, \\ E^{\text{tot}}(x) & = & \xi(x), & x \in C, \end{array}\right\} \tag{3}$$

  where S is an adequately small open neighborhood of L; the curve $C \subset S$ and the function ξ together define the desired near-field pattern.

- We have

$$(\Delta + k_0^2)E^{\text{tot}} = (k_0^2 - k(x)^2)E^{\text{tot}} = -\alpha\chi_L(x)E^{\text{tot}}, \; x \in S. \qquad (4)$$

- Decompose the total field E^{tot} in $\mathbf{R}^2 \setminus L$ into the sum $E^{\text{tot}} = E^{\text{inc}} + E^{\text{sca}}$ of an incident and a scattered field. Assume $|E^{\text{inc}}| \ll |E^{\text{sca}}|$ in $S \setminus \overline{L}$.
- Since the right-hand member of (4) is compactly supported, and since the scattered field must satisfy the Sommerfeld radiation condition in the plane, we have

$$\begin{aligned} E^{\text{sca}}(x) \approx E^{\text{tot}}(x) &= -\alpha\Phi_0 * (\chi_L E^{\text{tot}})(x) \\ &= -\alpha \int_{y \in L} \Phi_0(x-y)E^{\text{tot}}(y)dy, \quad x \in S \setminus \overline{L}, \end{aligned} \qquad (5)$$

and thus

$$E^{\text{inc}}(x) \approx E^{\text{tot}}(x) + \alpha \int_{y \in L} \Phi_0(x-y)E^{\text{tot}}(y)dy, \quad x \in S \setminus \overline{L}. \qquad (6)$$

Here $\Phi_0(x) = (i/4)H_0^{(2)}(k_0|x|)$ is the outgoing fundamental solution of the Helmholtz operator in the plane, and $H_0^{(2)}$ is the Hankel function of order zero and of the second kind.

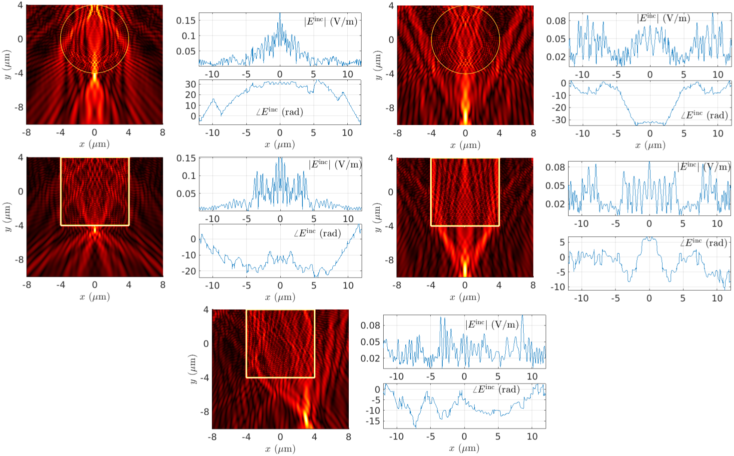


Figure: PNJ scanning achieved at the single optical wavelength $\lambda_0 = 532$ nm (common green laser). A 2D SiO_2 micro-lens with a circular cross-section of radius 4μm, or a square cross-section of side length 8μm, is illuminated along the negative y-axis by a computed structured incident field. The plots show the amplitude (in V/m) of the resulting total near field, normalized to maximum intensity of 1. Next to each near-field plot are the computed amplitude and phase profiles of the incident field that produce the desired total near field. The desired PNJ locations in μm are, from top to bottom: $(x, y) = (0, -4.532)$, $(x, y) = (0, -9.32)$, $(x, y) = (0, -4.532)$, $(x, y) = (0, -9.32)$ (radial shifts $1\lambda_0$ or $10\lambda_0$), and $(x, y) = (3.16, -9)$.

K., Scheel, Pedersen, and Hansen, *in review*.

Resolution of field control

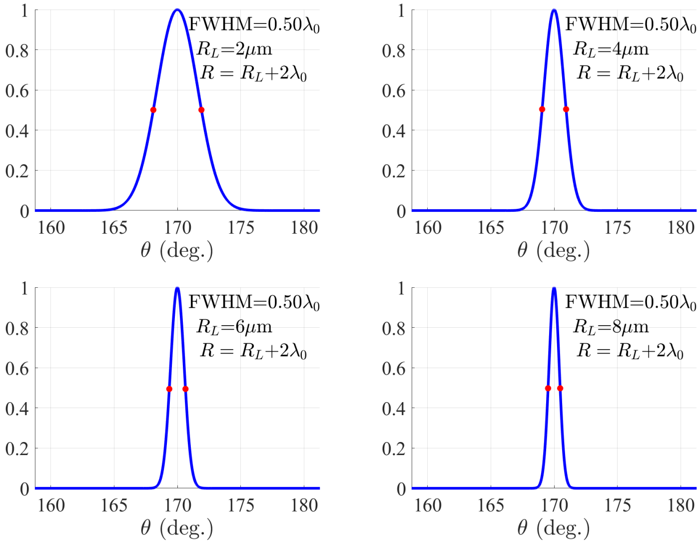


Figure: Desired PNJ profiles $|E^{\text{tot}}_{\text{PNJ}}(\theta)|$ at $\partial B_{R_L+\varrho}$ ($\times 10^5$ V/m) for different lens radii R_L.

$$F(s) \approx \sum_{\substack{m \in \boldsymbol{Z} \\ |m| \le \mathcal{B}}} \sigma_m (s, \psi_m)_{L^2(B_{R_L})} \phi_m \tag{7}$$

$$s_{\mathrm{TSVD}}^{\pm} = \sum_{\substack{m \in \boldsymbol{Z} \\ |m| \le \mathcal{B}_{\pm}}} \sigma_m^{-1} (E_{\mathrm{PNJ}}^{\mathrm{tot}}, \phi_m)_{L^2(\partial B_{R_L})} \psi_m, \tag{8}$$

$$\begin{aligned} E_s^{\mathrm{sca}\pm} = F(s_{\mathrm{TSVD}}^{\pm}) &\approx \sum_{\substack{m \in \boldsymbol{Z} \\ |m| \le \mathcal{B}_{\pm}}} \sigma_m (s_{\mathrm{TSVD}}^{\pm}, \psi_m)_{L^2(B_{R_L})} \phi_m \\ &= \sum_{\substack{m \in \boldsymbol{Z} \\ |m| \le \mathcal{B}_{\pm}}} (E_{\mathrm{PNJ}}^{\mathrm{tot}}, \phi_m)_{L^2(\partial B_{R_L})} \phi_m. \end{aligned} \tag{9}$$

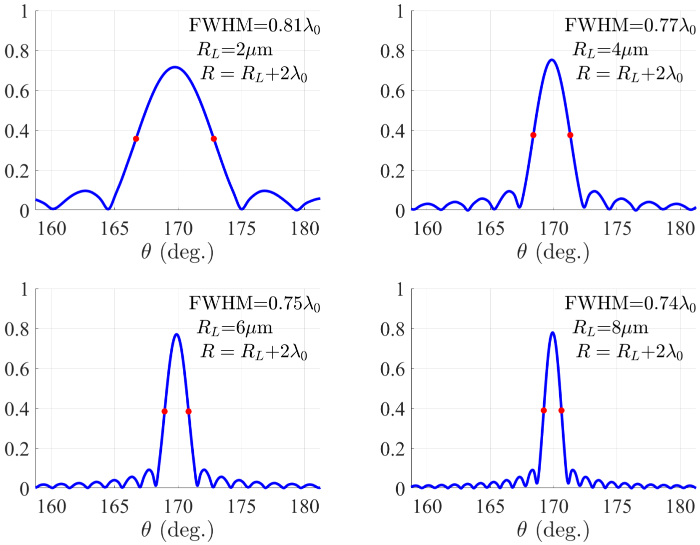


Figure: Resulting physically viable PNJ profiles $|E_s^{\mathrm{sca}}(\theta)|$ at $\partial B_{R_L+\varrho}$ ($\times 10^5$ V/m) for different lens radii R_L.

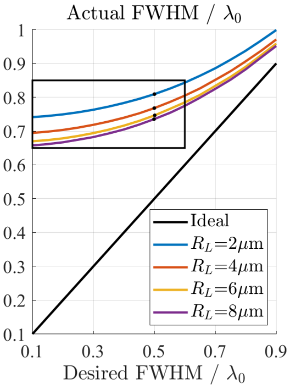


Figure: Waist-width prediction of PNJ profile for four different lens radii with a radial shift of $2\lambda_0$.

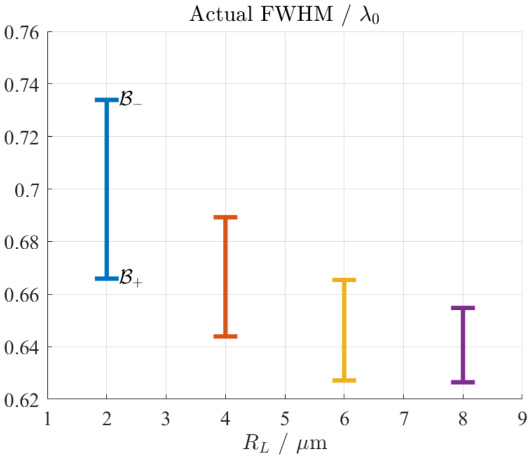
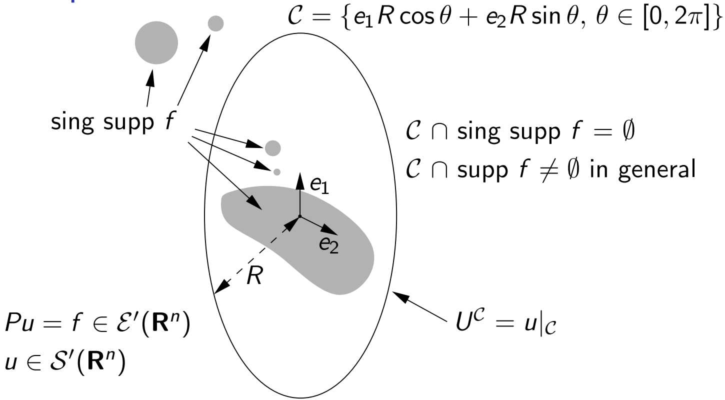


Figure: Angular PNJ resolution prediction using the projection from (9) with both bandwidth estimates $\mathcal{B}_\pm$ from K. (2018).

Problem setup



- K. & Winterrose (2021). arXiv:1912.10760v2
- $n \in \{2, 3, \dots\}$
- $\widehat{f}(\xi) = \int_{x\in\mathbf{R}^n} e^{-ix.\xi} f(x)$, $\xi \in \mathbf{R}^n$; $\quad \widehat{f} \in C^\infty(\mathbf{R}^n)$
- $\widehat{U}^{\mathcal{C}}_m := \int_{\theta=0}^{2\pi} e^{-im\theta} u(e_1 R\cos\theta + e_2 R\sin\theta)$
- **how does $\{\widehat{U}^{\mathcal{C}}_m\}_{m\in\mathbf{Z}}$ depend on $\widehat{f}$?** (relation to old work)

Comments

- no 'radiation condition'; uniqueness of solution of $Pu = f$ not guaranteed
- no singular value expansion of the forward operator
- Of all Euclidean spheres, only S^0, S^1 and S^3 admit a topological group structure. Therefore, it makes sense to define the Fourier transform of $u|_{S^{n-1}}$ only for $n = 1$, $n = 2$ and $n = 4$. For other dimensions n, one may pick specific bases of, say, $L^2(S^{n-1})$ and treat the projections of $u|_{S^{n-1}}$ onto the basis vectors as 'the Fourier coefficients of the measurement.' Instead, we choose to compute the Fourier coefficients of the measurement in terms of integrals over great circles for all dimensions n. Our analysis therefore estimates the magnitude of the spectral content of the measurement along any chosen direction in S^{n-1}.

The symbol p

Fix $\mu \in \mathbf{R}$, and let $p \in C^\infty(\mathbf{R}^n)$ be an elliptic symbol of Hörmander class $S^\mu(\mathbf{R}^n)$: for every $\alpha \in \mathbf{N}_0^n$ there is a constant C_α s.t.

$$|\partial^\alpha p(\xi)| \leq C_\alpha (1 + |\xi|^2)^{(\mu - |\alpha|)/2}, \quad \xi \in \mathbf{R}^n,$$

and there are positive constants C and R s.t.

$$|p(\xi)| \geq C(1 + |\xi|^2)^{\mu/2}, \quad |\xi| \geq R.$$

Assume furthermore that

$$p(\xi) = g(\xi) \prod_{j=1}^{N} (|\xi| - r_j)^{q_j}, \quad \xi \in \mathbf{R}^n,$$

where $N \in \mathbf{N}$, $0 < r_1 < r_2 < \cdots < r_N$, $q_j \in \mathbf{N}$, and $g \in C^\infty(\mathbf{R}^n \setminus \{0\})$ s.t. $|g(\xi)| \geq C_g > 0$; $|g(\xi)| \leq C'_g < \infty$ as $|\xi| \to 0$; and $|g(\xi)|$ at most polynomially increasing as $|\xi| \to \infty$.

The operator P

With $\mathcal{F}, \mathcal{F}^{-1} : \mathcal{S}'(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$ the (inverse) Fourier transform, and writing $\mathcal{F}u = \widehat{u}$, define $P : \mathcal{S}'(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$ by

$$(Pu)(\phi) = (\mathcal{F}^{-1} p\widehat{u})(\phi) = \widehat{u}(p\mathcal{F}^{-1}\phi), \quad u \in \mathcal{S}'(\mathbf{R}^n),\ \phi \in \mathcal{S}(\mathbf{R}^n),$$

that is, formally,

$$Pu(x) = (\mathcal{F}^{-1} p\widehat{u})(x) = (2\pi)^{-n} \int_{\xi \in \mathbf{R}^n} e^{ix.\xi} p(\xi)\widehat{u}(\xi),\ x \in \mathbf{R}^n,\ u \in \mathcal{S}'(\mathbf{R}^n).$$

- $P = \mathrm{Op}(p) \in \mathrm{OPS}^{\mu}(\mathbf{R}^n)$
- P is microlocal: if $f = Pu \in C^\infty$ in a neighbourhood of $x \in \mathbf{R}^n$ then $u \in C^\infty$ in a neighborhood of x.
- since p depends only on ξ, it is a multiplier, and P is a multiplier operator
- P is a Fourier (frequency)-domain filter
- a mapping $F : Pu \mapsto u|_{\mathcal{C}}$ might be expected to essentially invert the action of P in the Fourier domain

Admissible operators

- $P = \Delta + k^2$, $p(\xi) = k^2 - |\xi|^2 = -(|\xi| + k)(|\xi| - k)$, $g(\xi) = -(|\xi| + k)$, $N = 1$, $r_1 = k$, $q_1 = 1$.
- differential operators of the form $P = \sum_{j=0}^{M} c_j(-\Delta)^j$ with constants c_j such that at least one of the zeros of the polynomial $t \mapsto \sum_{j=0}^{M} c_j t^{2j}$ is positive
- pseudodifferential operators whose symbol $p(\xi)$ is independent of the base variable x and that can be transformed by a diffeomorphic pullback to a symbol with a radially symmetric zero set

The main results

- Fix $\rho > 0$ and let $\chi_\rho \in C_0^\infty(\mathbf{R}^n)$ be a window function s.t. $\chi(\xi) = 1$ for $|\xi| \leq \rho$ and $\chi(\xi) = 0$ for $|\xi| \geq 2\rho$.
- Define $\widehat{u}_\rho = \chi_\rho \widehat{u} \in \mathcal{E}'(\mathbf{R}^n)$; then u_ϱ is well-defined pointwise, with

$$u_\varrho(x) = (-2\pi)^{-n}\widehat{\widehat{u}}_\varrho(-x) = (-2\pi)^{-n}(\widehat{u}_\varrho)_\xi(e^{ix.\xi}), \quad x \in \mathbf{R}^n.$$

- Let

$$\widehat{U}^{\mathcal{C}}_{\rho,m} = \int_{\theta=0}^{2\pi} e^{-im\theta} u_\rho(e_1 R\cos\theta + e_2 R\sin\theta), \quad m \in \mathbf{Z}. \tag{10}$$

Lemma 1. (K. & Winterrose) $\lim_{\varrho\to\infty} \widehat{U}^{\mathcal{C}}_{\varrho,m} = \widehat{U}^{\mathcal{C}}_m$ for $m \in \mathbf{Z}$.

- Fix $d \in \mathbf{N}_0$ and assume $u \in \mathcal{S}'^d(\mathbf{R}^n)$, that is, $u \in \mathcal{S}'(\mathbf{R}^n)$ and there is a constant C satisfying

$$|u(\phi)| \leq C \sum_{|\alpha|\leq d} \sup_{x\in\mathbf{R}^n} (1+|x|^2)^{d/2}|\partial^\alpha \phi(x)|, \quad \phi \in \mathcal{S}(\mathbf{R}^n).$$

Theorem 1. (K. & Winterrose) If $Pu = f$ and $\widehat{f}$ has moderate growth then there are constants c' and, for any $\varrho > 0$, c_ϱ such that

$$\left|\widehat{U}^{\mathcal{C}}_{\varrho,m}\right| \leq c' + c_\varrho \sum_{j=1}^{N}\Big(\max\{1,|m|^{q_j},|m|^d\}|J_m(Rr_j)| + \max\{1,|m|^{q_j-1},\delta^{\mathrm{Kr}}_{d\geq1}|m|^{d-1}\}|J_{m+1}(Rr_j)|\Big) \quad \text{for } m \in \mathbf{Z}.$$

Comments

- "moderate growth": $\forall \alpha \in \mathbf{N}_0^n \; \exists C_\alpha,\, m_\alpha$ s.t. $\forall \xi \in \mathbf{R}^n$ $|\partial^\alpha \widehat{f}(\xi)| \leq C_\alpha (1 + |\xi|^2)^{m_\alpha/2}$; satisfied by, e.g., point sources
- The main information contained in Theorem 1 is the expected upper bound on the spectral location $|m|$ of the onset of rapid decay of the Fourier coefficients $\widehat{U}^{\mathcal{C}}_{\varrho,m}$; call this spectral location the 'bandwidth' of the measurement $U^{\mathcal{C}}$.
- Theorem 1 distinguishes between classes of solutions according to their order d as tempered distributions. The assumption $u \in \mathcal{S}'^d(\mathbf{R}^n)$ is a 'weak substitute' for a condition implying uniqueness. When uniqueness is ensured, d is given by the problem dimension n and the distributional order of f.
- The integral in (10) is the Funk-Radon transform of the integrand, evaluated at a single chosen direction $\nu \in S^{n-1}$ orthogonal to the plane of $\mathcal{C}$.

Lemma 1

- Fix $\rho > 0$ and let $\chi_\rho \in C_0^\infty(\mathbf{R}^n)$ be a window function s.t. $\chi(\xi) = 1$ for $|\xi| \leq \rho$ and $\chi(\xi) = 0$ for $|\xi| \geq 2\rho$.
- Define $\widehat{u}_\rho = \chi_\rho \widehat{u} \in \mathcal{E}'(\mathbf{R}^n)$; then u_ϱ is well-defined pointwise, with

$$u_\varrho(x) = (-2\pi)^{-n} \widehat{\widehat{u}}_\varrho(-x) = (-2\pi)^{-n} (\widehat{u}_\varrho)_\xi(e^{ix.\xi}), \quad x \in \mathbf{R}^n.$$

- Let

$$\widehat{U}^{\mathcal{C}}_{\rho,m} = \int_{\theta=0}^{2\pi} e^{-im\theta} u_\rho(e_1 R \cos\theta + e_2 R \sin\theta), \quad m \in \mathbf{Z}.$$

Lemma 1. (K. & Winterrose) $\quad \lim_{\varrho\to\infty} \widehat{U}^{\mathcal{C}}_{\varrho,m} = \widehat{U}^{\mathcal{C}}_m$ for $m \in \mathbf{Z}$.

Proof. Since $\chi_\rho \in C_0^\infty(\mathbf{R}^n)$ and $\chi_\rho(0) = 1$, we have $\lim_{\varrho\to\infty} \chi_\rho \phi = \phi$ in $\mathcal{S}(\mathbf{R}^n)$ for every $\phi \in \mathcal{S}(\mathbf{R}^n)$. Hence $\lim_{\varrho\to\infty} \widehat{u}_\varrho = \lim_{\varrho\to\infty} \chi_\rho \widehat{u} = \widehat{u}$ in $\mathcal{S}'(\mathbf{R}^n)$ with respect to its weak-$*$ topology. But $\mathcal{F}^{-1}$ is continuous on $\mathcal{S}'(\mathbf{R}^n)$, so $\lim_{\varrho\to\infty} u_\varrho = \lim_{\varrho\to\infty} \mathcal{F}^{-1}\widehat{u}_\varrho = u$ in $\mathcal{S}'(\mathbf{R}^n)$, and since u_ϱ and u are smooth in a neighborhood of $\mathcal{C}$, we have $\lim_{\varrho\to\infty} u_\varrho(x) = u(x)$ for every $x \in \mathcal{C}$.

An overview of the proof of Theorem 1

Assume $Pu = f$ with $u \in \mathcal{S}'(\mathbf{R}^n)$ and $f \in \mathcal{E}'(\mathbf{R}^n)$, $\widehat{f}$ of moderate growth. Since P is elliptic and $f \in C^\infty$ in a neighborhood of $\mathcal{C}$, the functions u_ρ and u are well-defined pointwise on $\mathcal{C}$. We have

$$\begin{aligned}
\widehat{U}^{\mathcal{C}}_{\rho,m} &= \int_{\theta=0}^{2\pi} e^{-im\theta} u_\rho(x(\theta)) = (-2\pi)^{-n} \int_{\theta=0}^{2\pi} e^{-im\theta} (\widehat{u}_\rho)_\xi (e^{i\xi.x(\theta)}) \\
&= (-2\pi)^{-n} (\widehat{u}_\rho)_\xi \left(\int_{\theta=0}^{2\pi} e^{-im\theta} e^{i\xi.x(\theta)} \right) \\
&= (-2\pi)^{-n} \widehat{u} \left(\chi_\rho(\xi) \int_{\theta=0}^{2\pi} e^{-im\theta} e^{i\xi.x(\theta)} \right).
\end{aligned}$$

Lemma. (K. & Winterrose)
Define

$$\mathcal{T}_m = \int_{\theta=0}^{2\pi} e^{-im\theta} e^{ia(X_1 \cos\theta + X_2 \sin\theta)},$$

where $a \in \mathbf{R} \setminus \{0\}$, $m \in \mathbf{Z}$, and where X_1 and X_2 are arbitrary complex constants. Writing J_m for the Bessel function of the first kind and integer order m, we have

$$\mathcal{T}_m = \begin{cases} 2\pi i^m J_m(a|X_1 + iX_2|) \exp -im\angle(X_1 + iX_2), & X_1 + iX_2 \neq 0, \\ 2\pi \delta^{\mathrm{Kr}}_{m=0}, & X_1 + iX_2 = 0. \end{cases}$$

Therefore, for any $u \in \mathcal{S}'(\mathbf{R}^n)$ satisfying $Pu = f$, we have

$$\begin{aligned}\widehat{U}^{\mathcal{C}}_{\rho,m} &= (-2\pi)^{-n}\widehat{u}\left(\chi_\rho(\xi)\int_{\theta=0}^{2\pi} e^{-im\theta}e^{i\xi.x(\theta)}\right)\\ &= (-1)^n(2\pi)^{1-n}i^m\widehat{u}\left(\chi_\rho(\xi)J_m(R|\xi.e_1 + i\xi.e_2|)e^{-im\angle(e_1.\xi+ie_2.\xi)}\right).\end{aligned}$$

Remark. It is a standard result that

$$J_m(Rr) = \sqrt{\frac{2}{\pi Rr}}\cos(Rr - (2m+1)\pi/4) + O((Rr)^{-3/2}), \quad m \in \mathbf{Z},$$

for $Rr \gg m^2$. Thus, if χ_ρ were omitted above, the tempered distribution $\widehat{u}$ would have to work on the function

$$J_m(R|\xi.e_1 + i\xi.e_2|)\exp -im\angle(e_1.\xi + ie_2.\xi),$$

which is not rapidly decaying. This illustrates the need for the cut-off function χ_ρ and for analyzing the approximate spectrum $\widehat{U}^{\mathcal{C}}_{\varrho,m}$, $\varrho > 0$.

Finding $\widehat{u}$

Since $Pu = f$ in $\mathcal{S}'(\mathbf{R}^n)$, we have equivalently $p\widehat{u} = \widehat{f}$ in $\mathcal{S}'(\mathbf{R}^n)$, where $\widehat{f}$ is of moderate growth. Also, $p \in C^\infty(\mathbf{R}^n)$ and $p\phi \in \mathcal{S}(\mathbf{R}^n)$ for every $\phi \in \mathcal{S}(\mathbf{R}^n)$, so if we can find $\mathfrak{p}^{-1} \in \mathcal{S}'(\mathbf{R}^n)$ such that $p\mathfrak{p}^{-1} = 1$ in $\mathcal{S}'(\mathbf{R}^n)$ then *one solution* of $p\widehat{u} = \widehat{f}$ in $\mathcal{S}'(\mathbf{R}^n)$ is $\widehat{u} = \widehat{f}\mathfrak{p}^{-1}$. Indeed, in that case

$$p \cdot \left(\widehat{f}\mathfrak{p}^{-1}\right)(\phi) = (p\mathfrak{p}^{-1})(\widehat{f}\phi) = \int_{\mathbf{R}^n} \widehat{f}\phi = \widehat{f}(\phi), \quad \phi \in \mathcal{S}(\mathbf{R}^n).$$

The corresponding Fourier coefficients at $\mathcal{C}$ are, for $m \in \mathbf{Z}$,

$$\boxed{\widehat{U}^{\mathcal{C}}_{\rho,m} = (-1)^n (2\pi)^{1-n} i^m \mathfrak{p}^{-1}\left(\widehat{f}(\xi)\chi_\rho(\xi) J_m(R|\xi.e1 + i\xi.e_2|)\exp(-im\angle(e_1.\xi + ie_2.\xi))\right).}$$

Note that $\mathfrak{p}^{-1}$ is the Fourier transform of a fundamental solution of P.

Finding $\mathfrak{p}^{-1}$

K. & Winterrose:

$$\begin{aligned} \mathfrak{p}^{-1}(\phi) &= \sum_{j=1}^{N}\sum_{k=1}^{q_j} c_{jk}\left[(r-r_j)_{+,\infty}^{-k} + (-1)^k (r-r_j)_{-,r_j}^{-k}\right] \otimes 1_{S^{n-1}}(r^{n-1}\phi/g) \\ &- \sum_{j=1}^{N}\sum_{k=n}^{q_j} c_{jk}\frac{(-1)^{k-n}\ln r_j}{(k-n)!}\delta_0^{(k-n)} \otimes 1_{S^{n-1}}(\phi/g) \end{aligned} \quad (11)$$

for $\phi \in \mathcal{S}(\mathbf{R}^n)$.

Also, $\mathfrak{p}^{-1} \in \mathcal{S}'^{\,\max\{q_j\}}(\mathbf{R}^n)$.

The distributions $(r - r_j)^{-k}_{\pm,\varrho}$

Hörmander I, Sec. 3.2: For complex a, define the functions

$$x_+^a = \begin{cases} x^a, & x > 0, \\ 0, & x \le 0, \end{cases} \quad \text{and} \quad x_-^a = \begin{cases} 0, & x \ge 0, \\ |x|^a, & x < 0. \end{cases}$$

If $\Re a > -1$ then x_+^a and x_-^a define distributions in $\mathcal{S}'(\mathbf{R})$.

Extension to all complex a by analytic continuation of the function $\mathbf{C} \ni a \mapsto \int_{x=0}^{\infty} x^a \phi(x)$, $\phi \in C_0^\infty(\mathbf{R})$, and by computing the residues at $a = -k$, $k \in \mathbf{N}$:

$$x_+^{-k}(\phi) = -\frac{1}{(k-1)!} \int_{x=0}^{\infty} (\ln x) \phi^{(k)}(x) + \frac{1}{(k-1)!} \phi^{(k-1)}(0) \sum_{j=1}^{k-1} j^{-1}, \quad \phi \in \mathcal{S}(\mathbf{R}),$$

$$x_-^{-k}(\phi) = -\frac{(-1)^k}{(k-1)!} \int_{x=0}^{\infty} (\ln x) \phi^{(k)}(-x) + \frac{(-1)^{k-1}}{(k-1)!} \phi^{(k-1)}(0) \sum_{j=1}^{k-1} j^{-1}, \quad \phi \in \mathcal{S}(\mathbf{R}).$$

The distributions $(r - r_j)^{-k}_{\pm,\varrho}$

We define the tempered distributions $r^{-k}_{\pm,\varrho}$, $k \in \mathbf{N}$, by

$$r^{-k}_{+,\varrho}(\phi) = -\frac{1}{(k-1)!}\int_{r=0}^{\varrho}(\ln r)\phi^{(k)}(r) + \frac{\phi^{(k-1)}(0)}{(k-1)!}\sum_{\nu=1}^{k-1}\frac{1}{\nu}$$
$$-\frac{1}{(k-1)!}\sum_{j=0}^{k-2}\phi^{(j)}(\varrho)\varrho^{-k+j+1}(k-j-2)!, \quad \phi \in \mathcal{S}(\mathbf{R}),$$

and

$$r^{-k}_{-,\varrho}(\phi) = r^{-k}_{+,\varrho}(\phi(-\cdot)) = -\frac{(-1)^k}{(k-1)!}\int_{r=0}^{\varrho}(\ln r)\phi^{(k)}(-r) + \frac{(-1)^{k-1}\phi^{(k-1)}(0)}{(k-1)!}\sum_{\nu=1}^{k-1}\frac{1}{\nu}$$
$$-\frac{1}{(k-1)!}\sum_{j=0}^{k-2}(-1)^j\phi^{(j)}(-\varrho)\varrho^{-k+j+1}(k-j-2)!, \quad \phi \in \mathcal{S}(\mathbf{R}),$$

respectively. Clearly, the distributions $r^{-k}_{\pm,\varrho}$ specialize to Hörmander's $r^{-k}_{\pm}$ when $\varrho = \infty$.

The distributions $(r - r_j)^{-k}_{\pm,\varrho}$

Now for every real b the mapping $\tau_b : \mathbf{R} \to \mathbf{R}$, $\tau_b(r) = r - b$, is smooth with surjective Jacobian $\tau_b'(r) = 1$, so (Hörmander I, Theorem 6.1.2) the pullback of $r^{-k}_{\pm,\varrho}$ by τ_b is given uniquely by

$$(r-b)^{-k}_{\pm,\varrho}(\phi) := \tau_b^* r^{-k}_{\pm,\varrho}(\phi) = r^{-k}_{\pm,\varrho}(\phi(\cdot + b)) = r^{-k}_{\pm,\varrho}(\phi \circ \tau_b^{-1}), \quad \phi \in \mathcal{S}(\mathbf{R}).$$

Finally, we write $\xi = r\omega$ for $\xi \in \mathbf{R}^n$, with $r \geq 0$ and $\omega \in S^{n-1}$, and let c_{jk} be the constants from the partial fraction decomposition

$$\prod_{j=1}^{N} (r - r_j)^{-q_j} = \sum_{j=1}^{N} \sum_{k=1}^{q_j} c_{jk} (r - r_j)^{-k}, \quad r \geq 0, \; r \neq r_j.$$

The trace $\widehat{U}^{\mathcal{C}}_{\rho,m}$

Lemma. (K. & Winterrose)
If $u = \mathcal{F}^{-1}(\widehat{f}\mathfrak{p}^{-1})$ then, for every positive ϱ and integer m,

$$(-1)^n(2\pi)^{n-1}i^{-m}\widehat{U}^{\mathcal{C}}_{\varrho,m} = \sum_{j=1}^{N}\sum_{k=1}^{q_j} c_{jk}\left[(r-r_j)_+^{-k} + (-1)^k(r-r_j)_{-,r_j}^{-k}\right] \tag{12}$$

$$\otimes 1_{S^{n-1}}\left(\frac{r^{n-1}\Psi_{\varrho,m}}{g}\right)$$

$$-\sum_{j=1}^{N}\sum_{k=n}^{q_j} c_{jk}\frac{(-1)^{k-n}\ln r_j}{(k-n)!}\delta_0^{(k-n)} \otimes 1_{S^{n-1}}(\Psi_{\varrho,m}/g), \tag{13}$$

where

$$\Psi_{\varrho,m}(r\omega) = \widehat{f}(r\omega)\chi(r\omega/\varrho)J_m(Rr|\omega.\widetilde{e}|)e^{-im\angle(\omega.\widetilde{e})},\ r \geq 0,\ \omega \in S^{n-1}, \tag{14}$$

and $\widetilde{e} = e_1 + ie_2$.
Corollary. (K. & Winterrose)
If $u = \mathcal{F}^{-1}(\widehat{f}\mathfrak{p}^{-1})$ then there is a constant c' and, for every positive ϱ, a constant c_ϱ such that

$$\left|\widehat{U}^{\mathcal{C}}_{\varrho,m}\right| \leq c' + c_\varrho\sum_{j=1}^{N}\left(\max\{1,|m|^{q_j}\}|J_m(Rr_j)| + \max\{1,|m|^{q_j-1}\}|J_{m+1}(Rr_j)|\right)$$

for $m \in \mathbf{Z}$.

Homogeneous solutions

Lemma. (K. & Winterrose)
If $u \in \mathcal{S}'^d(\mathbf{R}^n)$ satisfies $Pu = 0$ in $\mathcal{S}'(\mathbf{R}^n)$ then $\widehat{u} \in \mathcal{E}'^d(\mathbf{R}^n)$.

Define $\Phi : (0, \infty) \times S^{n-1} \to \mathbf{R}^n \setminus \{0\}$ by $\Phi(r, \omega) = r\omega$.

Lemma. (K. & Winterrose) If $u \in \mathcal{S}'^d(\mathbf{R}^n)$ satisfies $Pu = 0$ in $\mathcal{S}'(\mathbf{R}^n)$ then there are $u_{k,j} \in \mathcal{D}'^{d-k}(S^{n-1})$ such that

$$u(x) = \sum_{j=1}^{N} \sum_{k=0}^{d} \left(\partial_r^k \delta_{r_j}(r) \otimes u_{k,j}(\omega) \right) \left(e^{irx.\omega} \right), \quad x \in \mathbf{C}^n, \tag{15}$$

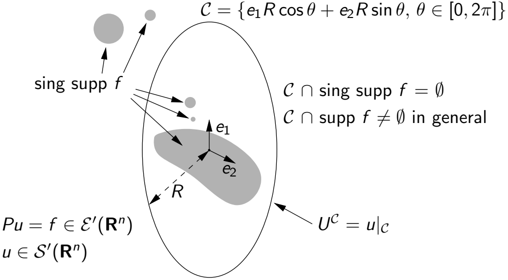
where

$$\sum_{k=q_j}^{d} \binom{k}{q_j} (-1)^k u_{k,j} = 0, \quad j \in \{1, \cdots, N\}. \tag{16}$$

Theorem. (K. & Winterrose) If $u \in \mathcal{S}'^d(\mathbf{R}^n)$ solves $Pu = 0$ in $\mathcal{S}'(\mathbf{R}^n)$ then there is a constant c satisfying

$$\left| \widehat{U}_m^{\mathcal{C}} \right| \leq c \sum_{j=1}^{N} \left(\max\{1, |m|^d\} |J_m(r_j R)| + \delta_{d \geq 1}^{\mathrm{Kr}} \max\{1, |m|^{d-1}\} |J_{m+1}(r_j R)| \right), \quad m \in \mathbf{Z}.$$

Example 1: the Helmholtz equation in $\mathbf{R}^n$ with point source



- $(\Delta + k^2)u = \partial_j^\nu \delta_y$ in $\mathbf{R}^n$, $n \in \{2, 3, \dots\}$, $j \in \{1, \dots, n\}$, $\nu \in \mathbf{N}_0$
- Sommerfeld radiation condition, so no nontrivial homogeneous solutions
- the unique outgoing fundamental solution:

$$\Phi_n(x) = \begin{cases} (-2\pi|x|)^{(1-n)/2}(2ik)^{-1}\partial_{|x|}^{(n-1)/2} e^{ik|x|}, & x \in \mathbf{R}^n \setminus \{0\},\ n \text{ odd}, \\ (-2\pi|x|)^{(2-n)/2}(4i)^{-1}\partial_{|x|}^{(n-2)/2} H_0^{(1)}(k|x|), & x \in \mathbf{R}^n \setminus \{0\},\ n \text{ even}, \end{cases}$$

- $u = (\partial_j^\nu \delta_y) * \Phi_n = (-1)^\nu (\partial_j^\nu \Phi_n)(\cdot - y) \in \mathcal{S}'^d(\mathbf{R}^n)$
- we need to estimate the order d of u

Example 1: the Helmholtz equation in $\mathbf{R}^n$ with point source

Lemma. (K. & Winterrose) $\Phi_n \in \mathcal{S}'^{(n+3)/2}(\mathbf{R}^n)$ for n odd. Furthermore, $\Phi_2 \in \mathcal{S}'^2(\mathbf{R}^2)$, $\Phi_4 \in \mathcal{S}'^3(\mathbf{R}^4)$ and $\Phi_n \in \mathcal{S}'^4(\mathbf{R}^n)$ for $n \in \{6, 8, 10, \dots\}$.

Remark. Our estimates of the distributional order in $\mathcal{S}'(\mathbf{R}^n)$ of outgoing fundamental solutions Φ_n coincide for $n = 1$ and $n = 2$; for $n = 3$ and $n = 4$; and for $n = 5$ and $n = 6, 8, 10, \dots$.

Corollary. (K. & Winterrose) $(\partial_j^\nu \Phi_n)(\cdot - y) \in \mathcal{S}'^{d(n)+\nu}(\mathbf{R}^n)$, with

$$d(n) = \begin{cases} (n+3)/2, & n \text{ odd}, \\ 2, & n = 2, \\ 3, & n = 4, \\ 4, & n \in \{6, 8, 10, \dots\}. \end{cases}$$

"Ground truth:"

$$\widehat{U}_m^{\mathcal{C}} = (-1)^\nu \int_{\theta=0}^{2\pi} e^{-im\theta} \partial_j^\nu \Phi_n(e_1 R\cos\theta + e_2 R\sin\theta - y), \quad m \in \mathbf{Z}.$$

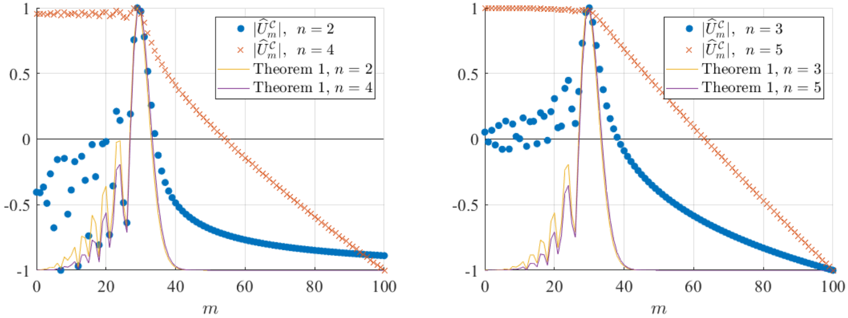
For every $\rho > 0$ we have

$$\widehat{U}_{\varrho,m}^{\mathcal{C}} = (-1)^\nu \int_{\theta=0}^{2\pi} e^{-im\theta} (\partial_j^\nu \Phi_n)_\varrho(e_1 R\cos\theta + e_2 R\sin\theta - y), \quad m \in \mathbf{Z},$$

and, by Theorem 1,

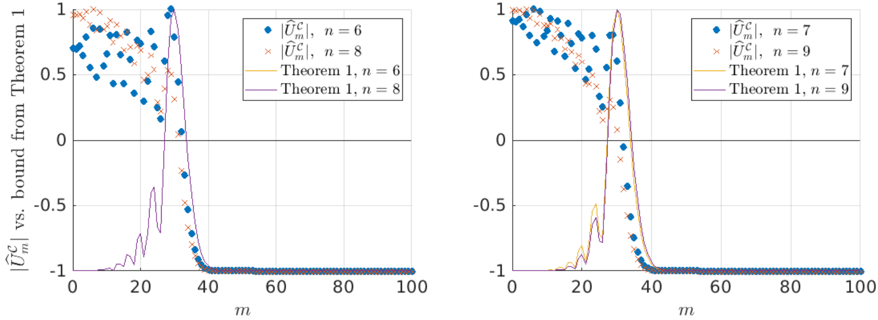
$$\left|\widehat{U}_{\varrho,m}^{\mathcal{C}}\right| \le c' + c_\varrho \times \begin{cases} \left(|m|^{(n+3+2\nu)/2} \cdot |J_m(kR)| + |m|^{(n+1+2\nu)/2} |J_{m+1}(kR)|\right), & n \text{ odd}, \\ \left(|m|^{2+\nu} \cdot |J_m(kR)| + |m|^{1+\nu} \cdot |J_{m+1}(kR)|\right), & n = 2, \\ \left(|m|^{3+\nu} \cdot |J_m(kR)| + |m|^{2+\nu} |J_{m+1}(kR)|\right), & n = 4, \\ \left(|m|^{4+\nu} \cdot |J_m(kR)| + |m|^{3+\nu} |J_{m+1}(kR)|\right), & n = 6, 8, 10, \ldots, \end{cases}$$

where c' and c_ϱ are (unknown) constants.



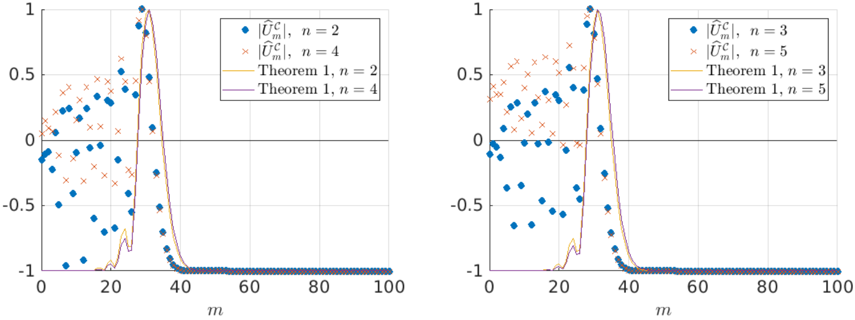
dimension n	bandwidth predicted by Theorem 1	actual bandwidth
2	29	29
3	30	30
4	30	29
5	30	29

K. & Winterrose (2021). Comparison of the actual spectrum $|\widehat{U}_m^{\mathcal{C}}|$ with the bound from Theorem 1. The spectrum and the bound are shifted and scaled to range in $[-1, 1]$. We are interested in the spectral location of the onset of rapid decay of $|\widehat{U}_m^{\mathcal{C}}|$. Parameters: $\nu = 0$, $R = 5.01$, $|y| = 5$.



dimension n	bandwidth predicted by Theorem 1	actual bandwidth
6	30	29
7	30	29
8	30	28
9	30	28

K. & Winterrose (2021). Comparison of the actual spectrum $|\widehat{U}_m^{\mathcal{C}}|$ with the bound from Theorem 1. The spectrum and the bound are shifted and scaled to range in $[-1, 1]$. We are interested in the spectral location of the onset of rapid decay of $|\widehat{U}_m^{\mathcal{C}}|$. Parameters: $\nu = 0$, $R = 5$, $y = (10, 0, \cdots, 0)$.



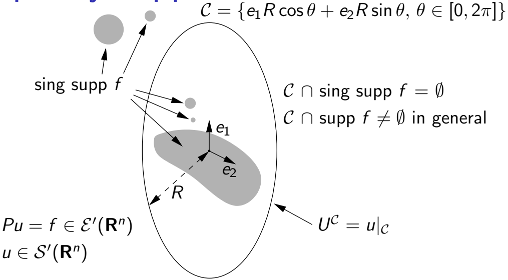
dimension n	bandwidth predicted by Theorem 1	actual bandwidth
2	31	29
3	31	29
4	31	29
5	31	29

K. & Winterrose (2021). Comparison of the actual spectrum $|\widehat{U}^{\mathcal{C}}_m|$ with the bound from Theorem 1. The spectrum and the bound are shifted and scaled to range in $[-1, 1]$. We are interested in the spectral location of the onset of rapid decay of $|\widehat{U}^{\mathcal{C}}_m|$. Parameters: $\nu = 5$, $R = 5$, $y = (10, 0, \cdots, 0)$.

"Bad" configurations

- $|y| \approx R$, $\nu = 0$, $n \in \{6, 7, 8, 9\}$ (a low-order point source in high dimension and close to the measurement circle $\mathcal{C}$)
- $|y| \approx R$, $\nu = 5$, $n \in \{2, 3, 4, 5\}$ (a high-order point source in low dimension and close to the measurement circle $\mathcal{C}$)
- consistent with the increasing severity of the singularity of the radiated field having an adverse effect on the numerical computations

Example 2: the Helmholtz equation in $\mathbf{R}^2$ or $\mathbf{R}^3$ with integrable compactly supported source

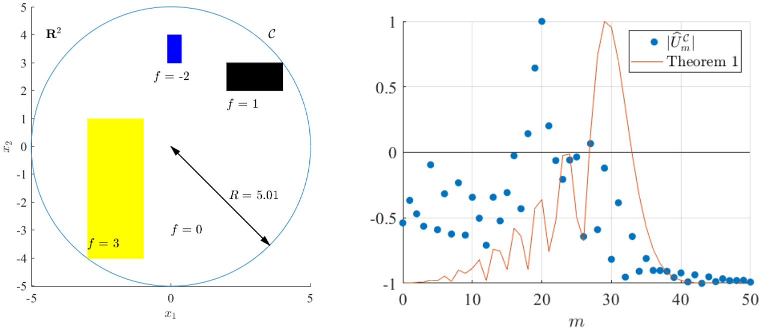


- $(\Delta + k^2)u = f \in \mathcal{E}'(\mathbf{R}^n) \cap L^p(\mathbf{R}^n)$, $n \in \{2, 3\}$, $p \in [1, \infty]$
- Sommerfeld radiation condition, so no nontrivial homogeneous solutions
- we get immediately that $f \in L^p_{\mathrm{loc}}(\mathbf{R}^n) \subseteq L^1_{\mathrm{loc}}(\mathbf{R}^n)$, hence $f \in \mathcal{E}'^0(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)$
- For any $\phi \in \mathcal{S}(\mathbf{R}^n)$ we have $f(-\cdot) * \phi \in \mathcal{S}(\mathbf{R}^n)$, so

$$|u(\phi)| = |(f * \Phi_n)(\phi)| = |\Phi_n(f(-\cdot) * \phi)| \leq C \sum_{|\alpha| \leq d(n)} \langle x \rangle^{d(n)} \sup_{x \in \mathbf{R}^n} |f * \partial^\alpha \phi|$$

$$\leq C \|f\|_{L^1(\mathbf{R}^n)} \sum_{|\alpha| \leq d(n)} \langle x \rangle^{d(n)} \sup_{x \in \mathbf{R}^n} |\partial^\alpha \phi(x)|,$$

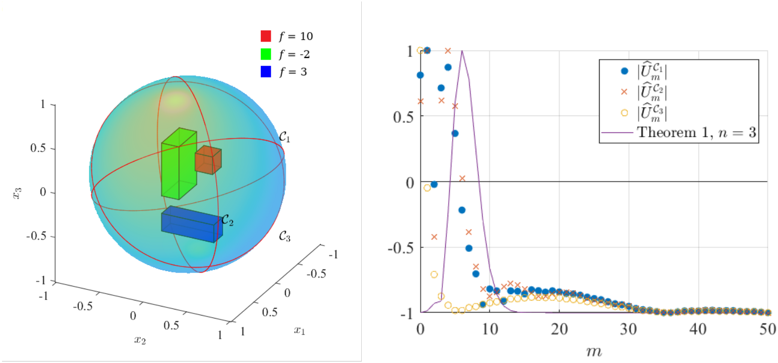
and consequently $u \in \mathcal{S}'^{d(n)}(\mathbf{R}^n)$. Thus **the same spectral cutoff estimates hold as for the point source case**.



K. & Winterrose (2021). A setup with a compactly supported, piecewise constant source in $\mathbf{R}^2$. Comparison of the actual spectrum $|\widehat{U}_m^{\mathcal{C}}|$ with the bound from Theorem 1. The spectrum and the bound are shifted and scaled to range in $[-1, 1]$. We are interested in the spectral location of the onset of rapid decay of $|\widehat{U}_m^{\mathcal{C}}|$. Parameters: $k = 2\pi$, supp $f \subset \{|x| \leq 5\}$. Theorem 1 predicts $\mathscr{B} = 29$, actual bandwidth is 27. Good correspondence with K. (2018), since

$$\{\operatorname{argmin}_{m \in \mathbf{N}_0}\{j_{m,1} \geq kR\}, \ldots, \operatorname{argmin}_{m \in \mathbf{N}_0}\{y_{m,1} \geq kR\}\} = \{26, \ldots, 29\}$$

for $kR = 2\pi \cdot 5.01$.



K. & Winterrose (2021). A setup with a compactly supported, piecewise constant source in $\mathbf{R}^3$. Comparison of the actual spectrum $|\widehat{U}_m^{\mathcal{C}}|$ with the bound from Theorem 1. The spectra and the bound are shifted and scaled to range in $[-1, 1]$. We are interested in the spectral location of the onset of rapid decay of $|\widehat{U}_m^{\mathcal{C}_j}|$ for $j = 1, 2, 3$. Parameters: $k = 2\pi$, $R = 1.01$, supp $f \subset \{|x| \leq 1\}$. Theorem 1 predicts $\mathscr{B} = 6$. The actual bandwidths are 4, 4 and zero, for measurement over $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$, respectively. Kirkeby, Henriksen, & K. (2020) predict

$$\{\text{argmin}_{m \in \mathbf{N}_0}\{j_{m+1/2,1} \geq kR\}, \dots, \text{argmin}_{m \in \mathbf{N}_0}\{y_{m+1/2,1} \geq kR\}\} = \{3, \dots, 5\}$$

for $kR = 2\pi \cdot 1.01$. The bandwidths hence appear similar for measurements **over the whole** S^2 and **along single great circles** of S^2.

Conclusion

- the onset of rapid decay in the measurement spectrum $\widehat{U}_m^{\mathcal{C}}$ seems to be uniform over a large class of multiplier equations $Pu = f$
- this onset depends on the structure of the zero set of the multiplier symbol, and on the distributional order of the solution $u \in \mathcal{S}'^d(\mathbf{R}^n)$, but not on other details of the symbol or, generally, on n
- the Bessel functions J_m dictate the non-asymptotic behavior of the measurement spectrum, and arise from our chosen geometry of the measurement set $\mathcal{C}$

References

G. Bao, J. Lin, and F. Triki, *A multi-frequency inverse source problem*, Journal of Differential Equations, 249 (2010), pp. 3443–3465.

R. Griesmaier, M. Hanke, and T. Raasch, *Inverse source problems for the Helmholtz equation and the windowed Fourier transform*, SIAM Journal on Scientific Computing, 34 (2012), pp. A1544–A1562.

R. Griesmaier, M. Hanke, and J. Sylvester, *Far field splitting for the Helmholtz equation*, SIAM Journal on Numerical Analysis, 52 (2014), pp. 343–362.

R. Griesmaier and J. Sylvester, *Far field splitting by iteratively reweighted l^1 minimization*, SIAM Journal on Applied Mathematics, 76 (2016), pp. 705–730.

R. Griesmaier and J. Sylvester, *Uncertainty principles for inverse source problems, far field splitting, and data completion*, SIAM Journal on Applied Mathematics, 77 (2017), pp. 154–180.

L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer, 2nd ed., 2003.

M. Karamehmedović and D. Winterrose, *On the transfer of information in multiplier equations*, to appear. arXiv:1912.10760v2

A. Kirkeby, M. T. R. Henriksen, and M. Karamehmedović, *Stability of the inverse source problem for the Helmholtz equation in $\mathbf{R}^3$*, Inverse Problems, 36 (2020), 055007.

M. Karamehmedović, *Explicit tight bounds on the stably recoverable information for the inverse source problem*, Journal of Physics Communications, 2 (2018), 095021.

M. Karamehmedović, A. Kirkeby, K. Knudsen, *Stable source reconstruction from a finite number of measurements in the multi-frequency inverse source problem*, Inverse Problems, 34 (2018), 065004.

R. Pierri and R. Moretta, *Asymptotic Study of the Radiation Operator for the Strip Current in Near Zone*, Electronics, 9 (2020).

R. Pierri and R. Moretta, *On the Sampling of the Fresnel Field Intensity over a Full Angular Sector*, Electronics, 10 (2021).

J. Xu and R. Janaswamy, *Electromagnetic Degrees of Freedom in 2-D Scattering Environments*, IEEE Transactions on Antennas and Propagation, 54 (2006), pp. 3882–3894.