Finite-time stabilization of some class of infinite dimensional systems

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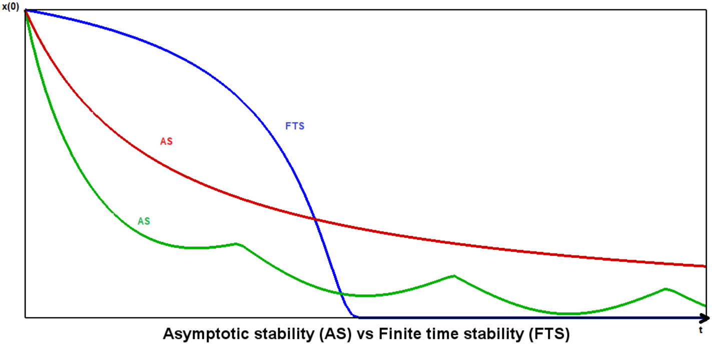
Outline

- Introduction.
- Finite time stabilisation (FTS).
  $\star$ The focus is put on two classes of infinite-dimensional systems : linear & bilinear systems.
  $\star$ Description of the approach : Lyapunov function (satisfying a differential inequality involving fractional powers) combined with sliding mode method.
- Applications to PDEs.

Introduction : Asymptotic stability vs FTS



Asymptotic stability (AS) vs Finite time stability (FTS)

Introduction : Asymptotic stability vs FTS

• The concept of stability (introduced by A.M. Lyapunov, 1892) is one of central notions of the control theory, and describes the system's response to small perturbations on initial conditions.
• The notion of asymptotic stability in control systems theory implies convergence of the system trajectories to a (Lyapunov stable) equilibrium state over an infinite horizon.
• This notion of asymptotic stability does not characterize a convergence time (i.e. the time response) of the system, and in such a case it is implicitly assumed that the equilibrium is reached when $t \to +\infty$.
• In many applications, however, it is desirable that the convergence of the system trajectories to the equilibrium state must do so in finite time rather than merely for $t \to +\infty$.

Introduction : Asymptotic stability of bilinear systems

Asymptotic stability of (BLS) in the literature

⋆ **Ball, J. M.,** & **Slemrod, M. (1979)**. Feedback stabilization of distributed semilinear control systems. Applied Mathematics and Optimization, 5(1), 169-179.

⋆ **Banks, S. P. (1986)**. Stabilizability of finite-and infinite-dimensional bilinear systems. IMA Journal of Mathematical Control and Information, 3(4), 255-271.

⋆ **Bounit, H.,** & **Hammouri, H. (1999)**. Feedback stabilization for a class of distributed semilinear control systems. Nonlinear Analysis : Theory, Methods & Applications, 37(8), 953-969.

⋆ **Berrahmoune, L. (2010)**. Stabilization of unbounded bilinear control systems in Hilbert space. Journal of mathematical analysis and applications, 372(2), 645-655.

Introduction : Asymptotic stability of bilinear systems

Asymptotic stability of (BLS) in the literature

$\star$ **Ammari, K., & Ouzahra, M. (2020)**. Feedback stabilization for a bilinear control system under weak observability inequalities. Automatica, 113, 108821.

$\star$ **Ammari, K., El Alaoui, S., & Ouzahra, M. (2021)**. Feedback stabilization of linear and bilinear unbounded systems in Banach space. Systems & Control Letters, 155, 104987.

$\star$ **Ouzahra, M. (2021)**. Exponential Stabilization of Unstable Bilinear Systems in Finite-and Infinite-Dimensional Spaces. IEEE Transactions on Automatic Control, 66(12), 5982-5989.

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Introduction : Considered systems

- Bilinear system :

$$(BLS) \qquad \dot{x}(t) = Ax(t) + u(t)Bx(t), \; x(0) = x_0 \in D(A).$$

- Linear system :

$$(LS) \qquad \dot{x}(t) = Ax(t) + Lv(t), \; x(0) = x_0 \in D(A).$$

- State space : H is a Hilbert space.
- The system operator A generates a quasi-contractive C_0-semigroup.
- The control operator B (of (BLS)) is bounded linear from H to H.
- The control operator L (of (LS)) is bounded linear from V (Hilbert space) to H.
- The functions $u(t) \in \mathbb{R}$ and $v(t) \in V$ are the controls.

Introduction : Notion of FTS

Definition

• The control system (BLS) (or (LS)) is said to be finite time stabilisable (at the origin) if
$\star$ (0) is a Lyapunov stable equilibrium,
$\star$ there is a feedback control for which the system in closed-loop admits a unique solution and
$\rightsquigarrow$ $\exists T = T(x_0) > 0$ s.t $x(t) = 0, \; \forall t \geq T$.
$\rightsquigarrow$ In that case, the time $T_* := \inf\{t > 0 : \; x(t) = 0\}$ is called settling time.

• There are other type of FTS : fixed/prescribed/uniform time stability.., for which we (eventually) need other type of (time-varying) feedback laws.

Introduction : FTS of finite-dimensional systems in the literature

• The notion of finite stability (also known today as finite-time stability, see **Bhat and Bernstein, 2000**) has been introduced in **Roxin, 1966**. Earlier, however (1951-1954), papers were published in the Russian literature (see e.g. **Erugin, 1951 ; Kamenkov G (1953) ; Lebedev A (1954)**).

• See **Dorato, P. (2006)** for further discussion on the history of FTS of finite dimensional systems (with corresponding literature).

Introduction : FTS of finite-dimensional systems in the literature

- **Erugin, N. (1951)**. "On the continuouation of solutions of differential equations" (in Russian . Prikl. Mat. Mekh. 17(4).
- **Kamenkov G (1953)**. On stability of motion over a finite interval of time (in Russian). Journal of Applied Math. and Mechanics (PMM) 17 :529-540.
- **Lebedev A (1954)**. The problem of stability in a finite interval of time (in Russian). Journal of Applied Math. and Mechanics (PMM) 18 :75-94.
- **Roxin, E. (1966)**. "On Finite Stability in Control Systems". Rendiconti del Circolo Matematico di Palermo. 15 : 273-283.
- **Bhat, S. P.,** & **Bernstein, D. S. (2000)**. Finite-time stability of continuous autonomous systems. SIAM Journal on Control and optimization, 38(3), 751-766.
- **Dorato, P. (2006)**. An overview of finite-time stability. Current trends in nonlinear systems and control, 185-194.

Introduction : FTS of infinite-dimensional systems in the literature

• For abstract infinite dimension systems,
⋆ the FTS has been mainly investigated for linear systems using nonlinear controls, and a few works have been addressed the FTS of infinite dimension bilinear systems.

⋆ There are many other works concerning the FTS of linear PDEs.

• See (**Efimov, D., & Polyakov, A. (2021)**) for a recent survey on existing results of finite-time stability with some motivating examples.

- **Coron, J. M., & Nguyen, H. M. (2017)**. Null controllability and finite time stabilization for the heat equations with variable coefficients in space in one dimension via backstepping approach. Archive for Rational Mechanics and Analysis, 225(3), 993-1023.
- **Song, Y., Wang, Y., Holloway, J., & Krstic, M. (2017)**. Time-varying feedback for regulation of normal-form nonlinear systems in prescribed finite time. Automatica, 83, 243-251.

Introduction : FTS of infinite-dimensional systems in the literature

- **Barbu, V. (2018)**. Controllability and stabilization of parabolic equations. Birkh$\ddot{a}$user.
- **Polyakov, A., Coron, J. M.,** & **Rosier, L. (2018)**. On homogeneous finite-time control for linear evolution equation in Hilbert space. IEEE Transactions on Automatic Control, 63(9), 3143-3150.
- **Steeves, D., Krstic, M.,** & **Vazquez, R. (2019, June)**. Prescribed-time H^1-stabilization of reaction-diffusion equations by means of output feedback. In 2019 18th European Control Conference (ECC) (pp. 1932-1937). IEEE.
- **Espitia, N., Polyakov, A., Efimov, D.,** & **Perruquetti, W. (2019)**. Boundary time-varying feedbacks for fixed-time stabilization of constant-parameter reaction-diffusion systems. Automatica, 103, 398-407.
- **Zhang, C. 2019**. Finite-time internal stabilization of a linear 1-D transport equation. Systems & Control Letters 133 (2019) 10452.

Introduction : FTS of infinite-dimensional systems in the literature

- **Holloway, J., & Krstic, M. (2019)**. Prescribed-time observers for linear systems in observer canonical form. IEEE Transactions on Automatic Control, 64(9), 3905-3912.
- **Sogore, M., & Jammazi, C. (2020)**. On the global finite-time stabilization of bilinear systems by homogeneous feedback laws. Applications to some PDE's. Journal of Mathematical Analysis and Applications, 486(2), 123815.
- **Efimov, D., & Polyakov, A. (2021)**. Finite-time stability tools for control and estimation.
- **Ouzahra, M. (2021)**. Finite-time control for the bilinear heat equation. European Journal of Control, 57, 284-293.

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Introduction : Notion of partial FTS

⋆ In control problems, there are often ultimate objectives other than the conventional stabilisation.

⋆ For example, one might be interested in making a subset of the state variables (as opposed to the full state) approach a desired set point. In this case, one deals with partial stability :

there is a given function $\xi(t) = Cx(t)$ (the output operator C is a mapping from H to an output space Y) and one wishes to make $\xi(t) = 0, t \geq T = T(x_0)$ using suitable control actions.

⋆ We then speak about partial stabilisation w.r.t C.

Introduction : Notion of partial FTS

$\star$ For a PDE with evolution domain Ω, on can be interested to the FTS of $Cy = y|_\omega$ for $\omega \subset \Omega$.

$\star$ Coupled system :

$$\dot{x}_1(t) = f_1(x_1, x_2),\ \dot{x}_2(t) = f_2(x_1, x_2),$$

one can consider the FTS of the part $Cx = x_1$ of the state $x = (x_1, x_2)$.

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$\star$ In addition of FTS of the part Cx of the state, one can look for further properties of the full state, for examples, the state may remain bounded in the time interval $[0, +\infty)$, or $x(t) \to l < \infty$ as $t \to +\infty, ..$

Introduction : Notion of partial FTS

⋆ **Jammazi, C. (2010).** On a sufficient condition for finite-time partial stability and stabilization : applications. IMA Journal of Mathematical Control and Information, 27(1), 29-56.

⋆ **Jammazi, C. (2014).** Continuous and discontinuous homogeneous feedbacks finite-time partially stabilizing controllable multichained systems. SIAM Journal on Control and Optimization, 52(1), 520-544.

⋆ **Wassim M. Haddad and Andrea L'Afflitto (2015)**. Finite-time partial stability and stabilization, and optimal feedback control. Journal of the Franklin Institute, 352(6), 2329-2357.

⋆ **K. Zimenko, D. Efimov, A. Polyakov and A. Kremlev.** "On Notions of Output Finite-Time Stability," 2019 18th European Control Conference (ECC), 2019, pp. 186-190, doi : 10.23919/ECC.2019.8796039.

⋆ **Zimenko, K., Efimov, D., Polyakov, A.,** & **Kremlev, A.** (2021). On necessary and sufficient conditions for output finite-time stability. Automatica, 125, 109427.

FTS of bilinear systems

Bilinear system :

$$(BLS) \qquad \dot{x}(t) = Ax(t) + u(t)Bx(t),\ x(0) = x_0.$$

$\star$ In the context of (abstract) infinite dimensional bilinear systems, asymptotic stability $(t \to +\infty)$ is more developed than FTS.

FTS of bilinear systems

FTS of (BLS) in the literature

- For abstract infinite dimension systems, the question of FTS has been treated in
  - $\star$ **Polyakov, A., Coron, J. M.,** $\&$ **Rosier, L. (2018)**,
  - $\star$ **Sogore** $\&$ **Jammazi, (2020)**,
- See also
  - $\star$ the book : **Polyakov, A. (2020)**,
  - $\star$ the survey : **Efimov, D.,** $\&$ **Polyakov, A. (2021)**.
- In the three last Refs., the FTS was studied for linear systems by using nonlinear feedback control, which is further applied to some bilinear examples.

FTS of bilinear systems

Building the FTS control

- In order to have an idea about the FTS feedback law and the Lyapunov function candidate, we formally differentiate

$$\frac{d}{dt}\|x(t)\|^2 = 2\langle Ax, x\rangle + 2u(t)\langle Bx, x\rangle$$

- so (at least for A dissipative) the dissipativness is guaranteed by taking $u(t) = -\langle Bx, x\rangle^{\alpha}$, for some choice of α, provided that :
  $\star$ this makes sense,
  $\star$ well-posedness : $\exists! x(\cdot)$,..
  $\star$ dissipativness : $\frac{d}{dt}\|x(t)\|^2 \leq 0$, i.e. $\langle Bx, x\rangle^{1+\alpha} \geq 0$,
- It is clear that FTS $\Rightarrow \alpha = -\mu$ with $\mu \in (0, 1/2)$.
- Let us discuss this, through some already treated situations !

FTS of bilinear systems

FTS : the heat equation

Let

$$(E_1) \quad \theta_t = \theta_{xx} + u(t)\theta$$

with DBC.

- It has been shown that (E_1) is FTS under the control

$$u(t) = -\|\theta(t)\|^{-\mu} = -\langle \theta(t), \theta(t) \rangle^{-\mu/2}, \ 0 < \mu < 1$$

(with $u(t) = 0$ in case of problems in defining it).
⋆ Estimation of the settling time $0 < T_* \leq \frac{\|\theta(0)\|^{\mu}}{\mu}$, (where $T_* := \inf\{t > 0 : \ \theta(t) = 0\}$).
⋆ The approach relies on the following (coercive) Lyapunov functions :
⇝ $V(x) = \|x\|_d^m, \ m > 0, \ \|\cdot\|_d$ is the homogeneous norm, (see **Polyakov, A., Coron, J. M., & Rosier, L. (2018)**).
⇝ $V(x) = \|x\|^2$ (see **Ouzahra, 2021**).

FTS of bilinear systems

Goal

- In the sequel, we will consider the (partial) FTS of the system (BLS) with $B = B^* \geq 0$.

- FTS of abstract bilinear system (BLS) with a coercive operator of control ($B \geq cI, \quad c > 0$), has been investigated by (**Sogore, M.,** $\&$ **Jammazi, C. (2020)**).

FTS of bilinear systems

A necessary condition

Theorem

If the system (BLS) is FTS, then

$$(OBS)$$

$$\forall \xi \in D(A); BS(t)\xi = 0, \forall t \geq 0 \Rightarrow \exists t_1 = t_1(x_0) > 0 \ / \ S(t_1)\xi = 0\cdot$$

⋆ A necessary condition for PFTS is :

$$(OBS)_C$$

$$\forall \xi \in D(A); \quad BS(t)\xi = 0, \ \ \forall t \geq 0 \ \ \Rightarrow \exists t_1 = t_1(x_0) > 0 \ / \ CS(t_1)\xi = 0\cdot$$

FTS of bilinear systems

Sufficient conditions for FTS : Description of the approach

- Build a feedback control u and a Lyapunov function candidate V s.t (in addition of the well-posedness) ;

$$\dot{V}(t) \leq -CV(t)^{-\mu}, \quad (*) \text{ with } 0 < \mu < 1$$

  so that for some $T = T(x_0) > 0$, one has

$$(**) \quad V(t) = 0, \; \forall t \geq T = T(x_0).$$

- In other word, the trajectory of the system is forced to move to the sliding surface $\Gamma : V(x) = 0$ and to stay in thereafter.
- The (positive) function V being not definite, one can not immediately deduce the extinction of the state.
- Then we will provide conditions for the sliding surface to be a zone of FTS, so that

  $(**) \Rightarrow$ FTS (extinction of the system in finite time) :

$$x(t) = 0, \; \forall t \geq T = T(x_0).$$

FTS of bilinear systems

Description of the approach : Some observations

- We (formally) have, for $t \in [0, T_*)$ (T_* is the settling time)

$$\frac{d}{dt}\|x(t)\|^2 = 2\langle x, Ax\rangle + 2u(t)\langle Bx, x\rangle$$

- Inspired by the case of heat equation, we can propose the following feedback law :

$$u(x) = -\langle Bx, x\rangle^{-\frac{\mu}{2}}\mathbf{1}_\Lambda,\ 0 < \mu < 1,$$

$\Lambda := \{x \in H : \langle Bx, x\rangle \neq 0\}$, which guarantees the dissipativness (at least for A dissipative) :

$$\frac{d}{dt}\|x(t)\|^2 \leq -2\langle Bx(t), x(t)\rangle^{1-\frac{\mu}{2}}.$$

FTS of bilinear systems

Description of the approach : Some observations

- Then two ideas raise for the construction of "Lyapunov function"

$$\exists c > 0 : \quad c \, \frac{d}{dt} \langle Bx, x \rangle \leq \frac{d}{dt} \|x(t)\|^2 \leq -2\langle Bx, x\rangle^{1-\frac{\mu}{2}}.$$

and

$$\frac{d}{dt} \|x(t)\|^2 \leq -2\langle Bx, x\rangle^{1-\frac{\mu}{2}} \leq c \, \|x(t)\|^\beta, \; c > 0, \text{ with appropriate } \beta.$$

- Thus we have the two following Lyapunov candidate functions :

$$V(x) = \langle Bx, x\rangle \text{ and } V(x) = \|x\|^2.$$

- Here, we use the first one (since the second choice leads to the situation of coercive control operator). Below

FTS of bilinear systems

Sufficient conditions : Main Assumptions :

$\star$ $(\mathcal{A}_1)$. $B^* = B \geq 0$ and $B^2 \geq \beta B, \ \beta > 0$.

$\star$ $(\mathcal{A}_2)$. For some $\omega \geq 0)$, we have

$$\langle Ax, x \rangle_B \leq \omega \|x\|_B^2, \ \forall x \in D(A),$$

where $\langle \cdot, \cdot \rangle_B = \langle B \cdot, \cdot \rangle$ and $\| \cdot \|_B = \langle B \cdot, \cdot \rangle^{1/2}$.

$\star$ $(\mathcal{A}_3)$. (Observability assumption) :

$$(OBS) \ \ \forall \xi \in D(A); \quad BS(t)\xi = 0, \ \ \forall t \geq 0 \ \ \Rightarrow \ \exists t_1 > 0 \ / \ S(t_1)\xi = 0\cdot$$

FTS of bilinear systems

Main Assumptions : Remarks and interpretations

$\star$ Assumption $(\mathcal{A}_1)$ is verified (in particular) in the following cases :

- coercive control operator B,
- wave equation,
- B is a (orthogonal) projection,
- if $B = B^* \geq 0$ is diagonalisable (e.g. B is compact), and take $\beta := \inf Sp(B) \setminus \{0\}$, i.e. the first non null eigenvalue of B,..

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FTS of bilinear systems

$\star$ Assumption $(\mathcal{A}_2)$ means that the operator A is quasi-dissipative w.r.t $\langle\cdot,\cdot\rangle_B$.

$\star$ Assumption $(\mathcal{A}_3)$: The assumption (OBS) holds if the (weak) final observability condition on some $[0, t_1]$ holds (in the sense of linear system).

$\star$ Moreover, if $S(t)$ is one to one (for some $t > 0$), then (OBS) reads as follows :

$$\forall \xi \in H; \quad BS(t)\xi = 0, \ \ \forall t \geq 0 \ \ \Rightarrow \xi = 0,$$

(which is guaranteed by the (weak) initial observability condition).

FTS of bilinear systems

FTS result.

Theorem

⋆ *Under the assumptions* $(\mathcal{A}_1) - (\mathcal{A}_3)$, *the control law*

$$u(x) = -\left(\frac{\omega}{\beta} + \langle Bx, x\rangle^{-\frac{\mu}{2}}\right)\mathbf{1}_{\Lambda}, \ (0 < \mu < 1)$$

stabilises the system (BLS) *in finite time.*

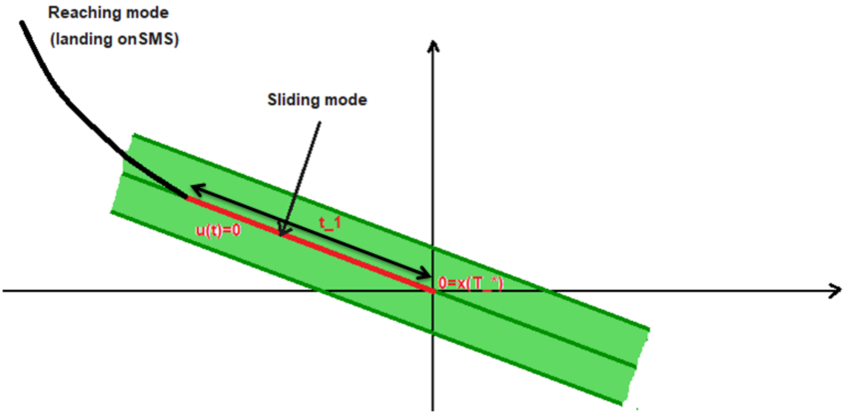
⋆ *Furthermore, the settling time is s.t.*

$$T_* \leq \frac{\langle Bx_0, x_0\rangle^{\frac{\mu}{2}}}{\beta\mu} + t_1(x_0).$$

Remark.

⋆ If $(\mathcal{A}_3)$ is replaced by $(OBS)_C$, then we have PFTS with the same control and the same estimation of the settling time (where in that case t_1 may depend on C).

Sliding mode and FTS



FTS of bilinear systems

Remarks.

⋆ If the semigroup is one to one, then $T_* \leq \frac{\langle Bx_0, x_0\rangle^{\frac{\mu}{2}}}{\beta\mu}$.
In this case, the operator is one to one as well.

⋆ If in assumptions $(\mathcal{A}_1)-(\mathcal{A}_2)$, we have $\langle Ax, x\rangle_B = 0,\ \forall x \in D(A)$ and $B^2 = \beta\ B,\ \beta \in \mathbb{R}^*$, then the estimate of the settling time is optimal, i.e $T_* = \frac{\langle Bx_0, x_0\rangle^{\frac{\mu}{2}}}{\beta\mu} + t_1$. If t_1 is continuous w.r.t initial states (e.g if $S(t)$ one to one or nilpotent semigroup), then so is T_*.

⋆ In assumption $(\mathcal{A}_2)$, one can replace the quasi-dissipativness of A w.r.t $\langle\cdot,\cdot\rangle_B$, by the existence of a function $f : H \to R$ s.t $\frac{\langle Ax, x\rangle_B}{\|x\|_B^2} \leq f(x),\ \forall x \in D(A)$ and $F := f\ B$ is a Lipschitz function.

⋆ In case of PFTS, we further have some information about the full state : $\|x(t)\| \leq \|x_0\|$ (in case of $\omega \neq 0$, we use the change of variable $z(t) = e^{-t\omega}x(t)$) and $\|x(t)\| \to l,\ t \to +\infty$ with $0 \leq l \leq \|x_0\|$.

FTS of linear systems

Linear system

$$(LS) \qquad \dot{x}(t) = Ax(t) + Lv(t), \; x(0) = x_0.$$

- State space : H is a Hilbert space.
- The system operator A generates a quasi-contractive C_0-semigroup on the Hilbert space H.
- The control operator L is bounded linear from V (Hilbert space) to H.
- The function $v(t) \in V$ is the control.

FTS of the linear system

⋆ In order to use the bilinear approach, we look for a nonlinear control law of the form :

$$v(x) = u(x)L^*x$$

where u is a new (bilinear) control.

⋆ The resulting closed-loop equation is

$$\dot{x}(t) = Ax(t) + u(x(t))LL^*x(t).$$

⋆ Then the FTS control candidate (take $B = LL^*$ in (BLS)) will be :

$$v(t) = \begin{cases} -\frac{\omega}{\beta} - \frac{L^*x(t)}{\|L^*x(t)\|^{\mu}} & \text{if } L^*x(t) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $0 < \mu < 1$.

FTS of the linear system

Corollary

Assume that the operator L is such that the assumption $(\mathcal{A}_1) - (\mathcal{A}_3)$ holds for $B = LL^*$. Then the system (LS) is finite-time stable if and only if the following condition holds :

For any $y \in D(A), L^* S(t) y = 0, \ \forall t \geq 0 \Rightarrow \exists t_1 > 0 \ s.t \ S(t_1) y = 0.$

Remark.

⋆ The PFTS w.r.t C of the system (LS) is equivalent to :
For any $y \in D(A), L^* S(t) y = 0, \ \forall t \geq 0 \Rightarrow \exists t_1 > 0 \ s.t \ CS(t_1) y = 0.$

⋆ Note that the necessity of the observability condition above holds even for a non quasi-contractive semigroup.

Further FTS results : the case of multiple equilibrium

⋆ In the previous results, the equilibrium is unique due to the observation condition. In the absence of the observation condition, the equilibrium set may contain several elements.

⋆ Given a (nonlinear) semigroup $T(t)$ with generator $\mathcal{A}$, we denote by $E_{\mathcal{A}}$ the set of equilibrium states given by

$$E_{\mathcal{A}} = \mathcal{A}^{-1}(0)$$

⋆ It is easy to see that : $E_{\mathcal{A}} = \{p \in H : \ T(t)p = p, \ \forall t \geq 0\}$.
⋆ In particular, the set of equilibrium states of the uncontrolled system is given by :

$$E_A = \{p \in D(A) : \ Ap = 0\} = \{p \in H / \ S(t)p = p, \ \forall t \geq 0\}.$$

⋆ We will provide necessary and sufficient conditions s.t the solution of every system in closed-loop satisfies $x(t) = p, \ \forall t \geq T > 0$, for some equilibrium p of the system at hand. In other words, every trajectory may go to some equilibrium point.

Further results for FTS of bilinear systems

⋆ Let $M = \{x \in H / \ BS(t)x = 0, \ \forall t \geq 0\}$. We have the following result.

Theorem

⋆ *Let assumptions* $(\mathcal{A}_1) - (\mathcal{A}_2)$ *hold.*
⋆ *Then*

$$M \subset \{z/\exists t_1 > 0; S(t_1)z \in E_A\} \Leftrightarrow$$

$$(\forall x_0 \in H)\,(\exists T_1 > 0); x(t) = p, t \geq T_1 \ \textit{ for some } \ p \in E_{\mathcal{A}}.$$

⋆ If $S(t)$ is one to one, then $\{z/ \ \exists t_1 > 0; \ S(t_1)z \in E_A\} = E_A$.

Further results for FTS of bilinear systems

⋆ The asymptotic version of this result has been considered by (**Pazy, 1978**) for nonlinear semigroup and by (**Berrahmoune, 2010**) for bilinear systems.

⋆ **Pazy, A. (1978).** On the asymptotic behavior of semigroups of nonlinear contractions in Hilbert space. Journal of Functional Analysis, 27(3), 292-307.

⋆ **Berrahmoune, L. (2010).** Stabilization of unbounded bilinear control systems in Hilbert space. Journal of Mathematical Analysis and Applications, 372(2), 645-655.

Further FTS results of the linear system

⋆ We now give a linear version of the last theorem, then we show that some of the assumptions above are satisfied.

⋆ We look for a control $v(x) = v_1(x) + u(x)\langle x, g\rangle\theta$ with $g = L\theta$, where v_1 is a linear control that will guarantee $(\mathcal{A}_2)$.

⋆ Implementing the control law $v(x)$ in the linear system, we get a bilinear system with control operator : $Bx = \langle x, g\rangle g$, so that $(\mathcal{A}_1)$ is also fulfilled.

Further FTS resuls of the linear system

Following the same techniques, we can prove the following result.

Theorem

Assume that there exists $g \in D(A^*)$ *such that* $L\theta = g$ *for some* $\theta \in V$ *and let* $Cx = \langle x, g \rangle$, *and let* $A_g = A - \langle \cdot, A^* g \rangle \frac{g}{\|g\|}$ *with* $D(A)$ *as domain.*

- *Then,*

⋆ *the system (LS) is PFTS w.r.t* C *by the control*

$$v(t) = \begin{cases} -\left(\frac{\langle x(t), A^* g\rangle}{\|g\|^2} + |\langle x(t), g\rangle|^{-\mu}\right)\theta & \text{if } \langle x(t), g\rangle \neq 0 \\ -\frac{\langle x(t), A^* g\rangle}{\|g\|^2}\theta & \text{otherwise} \end{cases}$$

⋆ *Moreover, under the following condition*

$$\langle S_g(t)z, g\rangle = 0, \ \forall t \geq 0 \Rightarrow \exists t_1 > 0; \ S_g(t_1)z \in E_{\mathcal{A}_g}$$

we have $x(t) = p, \ t \geq T_1 > 0$ *for some equilibrium point* p *of the system in closed-loop, where* $\mathcal{A}_g(x) := A_g(x) + u(x)Bx, x \in D(A)$.

Applications : Heat equation

$\star$ Let Ω be an open bounded set of $\mathbb{R}^N$.

• **Situation 1.**

Let us consider the following reaction diffusion bilinear system

$$\begin{cases} y_t(x,t) = \Delta y(x,t) + u(t)\phi(x)y(x,t) & (x,t) \in \Omega \times (0,+\infty) \\ y(x,t) = 0 & (x,t) \in \partial\Omega \times (0,+\infty) \\ y(.,0) = y_0 \end{cases}$$

where $\phi \in L^\infty(\Omega)$ and $\phi(x) > 0$ a.e. $x \in \Omega$.

$\star$ For all $y_0 \in H$ the above system is finite time stable at a settling time T^* under the following control

$$u(t) = \begin{cases} -\|\sqrt{\phi(.)}y(t)\|^{-\mu} & \text{if } y(t) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$0 < \mu < 1.$

Applications : Heat equation

Remarks.

⋆ The mono-dimensional case has been studied in (Coron et al. 2018) for $\phi(x) \geq cx^2,$ a.e. on $\Omega := (0,1)$.

⋆ This FTS result also holds for other situations, e.g, multidimensional case, and $\Delta + a(x)I$ (with $a \in L^\infty(\Omega)$) instead of Δ.

⋆ The result remains true for NBC.

⋆ We also have the FTS of the corresponding linear equation :

$$y_t(x,t) = \Delta y(x,t) + v(t)\phi.$$

Applications : Heat equation

• **Situation 2.**
Consider the bilinear heat equation with NBC.

$$Bx = x - \langle x, \mathbf{1} \rangle \mathbf{1}$$

$\star$ Here, the (initial) observation assumption (OBS) does not hold, so we do not expect to have the FTS at the origin.

$\star$ The assumptions $(\mathcal{A}_1) - (\mathcal{A}_2)$ are verified.

$\star$ We have $E_{\mathcal{A}} = \ker A \cap \ker B =$ span $\mathbf{1}$.

$\star$ Then for every $y_0 \in H^2(0,1)$ s.t. $y_0'(0) = y_0'(1) = 0$, we have $\exists\, T_1 > 0$;

$$y(t) = c\,\mathbf{1},\ \forall t \geq T_1 \text{ for some } c \in \mathbb{R}.$$

Applications : Transport equation

Let us consider the following transport system with internal control

$$\begin{cases} y_t + y_x + a(x)y = v(t)\chi_\omega & (x,t) \in \Omega \times (0,+\infty) \\ y(0,t) = 0 & t \in \times(0,+\infty) \\ y(.,0) = y_0 \end{cases}$$

where $\omega = (0,\alpha) \subset \Omega = (0,L),\ 0 < L < +\infty,\ a \in L^\infty(\Omega)$. Here, χ_ω indicates the characteristic function of ω.

$\star$ The function $y(t) := y(\cdot,t)$ is the state and $v(t) := v(\cdot,t) \in L^2(\Omega)$ is the additive control.

Applications : Transport equation

$\star$ Here, the assumptions $(\mathcal{A}_1) - (\mathcal{A}_3)$ are all verified, hence the full state is FTS.

$\star$ The linear FTS control is given by

$$v(t) = \begin{cases} -\big(\int_0^\alpha |y(x,t)|^2 dx \big)^{-\mu/2} \chi_\omega y(t) & \text{if } \chi_\omega y(t) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$\star$ Equivalently, the following control

$$u(t) = \begin{cases} -\big(\int_0^\alpha |y(x,t)|^2 dx \big)^{-\mu/2} & \text{if } \chi_\omega y(t) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

guarantees the FTS of the bilinear version.

Remark. As we can see for $a = 0$, the zero control also guarantees the FTS at $T \geq 1$ so that $T_* \leq 1$. However, with the control u (or v), the estimate of the settling time is better than 1 for some initial states.

PFTS : Wave equation

⋆ Bilinear wave equation evolving in $\Omega = (0,1)$

$$y_{tt} = \Delta y + u(t)\langle y_t, \mathbf{1}\rangle \mathbf{1}$$

with NBC. Let $z(t) = (y(t), \dot{y}(t))$.
⋆ We have

$$\forall y = (a,b) \in H^1(\Omega) \times L^2(\Omega),\ BS(t)y = 0,\ \forall t \geq 0 \Rightarrow \langle a, \mathbf{1}\rangle = \langle b, \mathbf{1}\rangle = 0$$

⋆ Then we have the PFTS w.r.t $C(a,b) = \begin{pmatrix} \langle a, \mathbf{1}\rangle \\ \langle b, \mathbf{1}\rangle \end{pmatrix}$,

$$\exists T_1 > 0 \ \ s.t \ \ \langle y(t), \mathbf{1}\rangle = \langle y_t(t), \mathbf{1}\rangle = 0,\ \forall t \geq T_1.$$

⋆ Moreover, we have $\|z(t)\| \to \|z_0\|$, as $t \to +\infty$.

⋆ **Remark.**
⋆ Here, we have $B^2 = B$ and $\langle Ax, x\rangle_B = 0, x \in D(A)$, so the estimate of the settling time is optimal and depends continuously on the initial state.
⋆ This remains true for the N-dimensional wave equation.

Remarks

⋆ While the additive FTS control is continuous near the equilibrium, the bilinear FTS control is not, however this is compensated when it enters the system in a multiplicative way (by the state).

Remarks

$\star$ In a joint work with Jammazi and Sogoré (submitted, 2021), the following issue are considered :

- FTS of bilinear systems with a coercive control operator,

- PFTS in prescribed time of the heat equation using time-varying control with $Cx = x|_\omega$ (ω being a subregion of the evolution domain Ω), that is for any a priori given time $T > 0$, there is a (time-varying) feedback law for which $x(t)|_\omega \to 0$, as $t \to T^-$.

- PFTS in prescribed tim of the wave equation using a time-varying feedback control with $C(x, \dot{x}) = \dot{x}$, i.e. $\dot{x}(t) \to 0$, as $t \to T^-$.

Thank you