Delocalization on quantum trees

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Plan

1. Dynamics and spectra - some general theory.
2. Quantum trees.
3. Open problems.
4. Some proof ideas.

Dynamics and spectra
some general theory

Spectrum

For a closed operator H on a Hilbert space $\mathscr{H}$,

$$\sigma(H) = \{\lambda \in \mathbb{C} : H - \lambda I \text{ has no bounded inverse}\}.$$

$$\{\text{eigenvalues}\} =: \sigma_p(H) \subseteq \sigma(H).$$

$$\sigma_p(H) = \sigma(H) \qquad \text{if } \dim \mathscr{H} < \infty$$

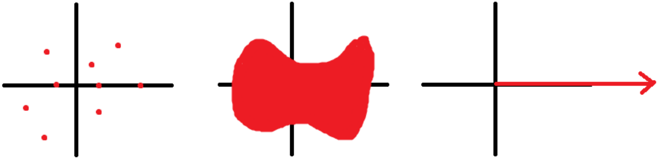

Figure – Spectrum of a matrix (left), bounded operator (middle), unbounded self-adjoint operator (right).

Spectrum

Spectral Theorem : Let H be self-adjoint on $\mathscr{H}$ and $\phi \in \mathscr{H}$. Then there exists μ_ϕ^H such that for any bounded Borel $f : \sigma(H) \to \mathbb{C}$,

$$\langle \phi, f(H)\phi \rangle = \int_{\sigma(H)} f(\lambda)\, \mathrm{d}\mu_\phi^H(\lambda)$$

Lebesgue decomposition :

$$\mu = \mu_{ac} + \mu_{sc} + \mu_{pp}$$

$$\mathscr{H}_{xx} = \{\phi \in \mathscr{H} : \mu_\phi \text{ is } xx\}$$

$$\mathscr{H} = \mathscr{H}_{ac} \oplus \mathscr{H}_{sc} \oplus \mathscr{H}_{pp}$$

Spectrum

$$\sigma_{ac}(H) = \sigma(H|_{\mathscr{H}_{ac}}) , \qquad \sigma_{sc}(H) = \sigma(H|_{\mathscr{H}_{sc}}) , \qquad \sigma_{pp}(H) = \sigma(H|_{\mathscr{H}_{pp}}) .$$

Maximal spectral measure : We may combine a countable sequence of μ_{ϕ_n} into a *maximal spectral measure* ρ^H,

$$\sigma_{xx}(H) = \operatorname{supp} \rho^H_{xx}$$

Borel transform : for $z \in \mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$,

$$F_\mu(z) = \int_{\mathbb{R}} \frac{1}{t - z} \, \mathrm{d}\mu(z) .$$

Spectrum

Define

$$M = \{\lambda \in \mathbb{R} : \operatorname{Im} F_\mu(\lambda + \mathrm{i}0) > 0\}\,,$$

$$M_{ac} = \{\lambda \in \mathbb{R} : 0 < \operatorname{Im} F_\mu(\lambda + \mathrm{i}0) < \infty\}$$

Then [1]

$$\operatorname{supp} \mu = \overline{M}\,, \qquad \operatorname{supp} \mu_{ac} = \overline{M_{ac}}^{ess}\,.$$

$$F_{\mu_\phi}(z) = \langle \phi, (H - z)^{-1} \phi \rangle\,.$$

$\rightsquigarrow$ need resolvent control as $\operatorname{Im} z \downarrow 0$.

1. Here $\overline{N}^{ess} = \{\lambda \in \mathbb{R} \mid \forall \varepsilon > 0 : |N \cap (\lambda - \varepsilon, \lambda + \varepsilon)| > 0\}$ is the *essential closure* of N. One checks that $\overline{N}^{ess}$ is closed and $\overline{N}^{ess} \subseteq \overline{N}$. But in general $N \not\subset \overline{N}^{ess}$, as isolated points of N will disappear.

Dynamics

Evolution semigroup : $\mathrm{e}^{-\mathrm{i}tH}$.

If H bounded, $\mathrm{e}^{-\mathrm{i}tH} = \sum_{k=0}^{\infty} \frac{(-\mathrm{i}tH)^k}{k!}$, otherwise spectral theorem.

If $\psi_0 \in \mathscr{H}$ is some initial state of the system, then $\psi_t := \mathrm{e}^{-\mathrm{i}tH}\psi_0$ describes the state of the system at time t.

$\mathrm{e}^{-\mathrm{i}tH}$ is unitary $\rightsquigarrow \|\mathrm{e}^{-\mathrm{i}tH}\psi\| = \|\psi\|$ for all time.

Dynamics

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \implies \mathrm{e}^{-\mathrm{i}tA} = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}ta} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}tb} \end{pmatrix} \qquad \text{(multiplication op.)}$$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \implies \mathrm{e}^{-\mathrm{i}tA} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \qquad \text{(rotation op.)}$$

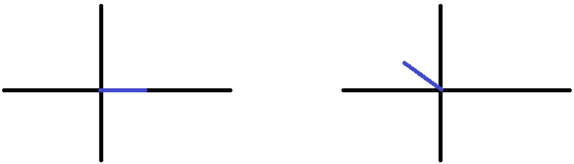

Figure – $\psi_0 = e_1$ (left), $\psi_t = \mathrm{e}^{-\mathrm{i}tA} e_1$ (right).

Dynamics

Main question : large time behavior of $\mathrm{e}^{-\mathrm{i}tH}\psi_0$.

Say on $L^2(\mathbb{R}^d)$ with $H = -\Delta + V$.

Stays in a compact set for all t ?

Escapes from any given compact set after some time ?

Figure – ψ_0 (left), $\psi_t = \mathrm{e}^{-\mathrm{i}tA}\psi_0$ (right).

Dynamics and spectra

RAGE (69-78) :

$$f \in \mathscr{H}_p \iff \forall \varepsilon > 0\ \exists K \subset \mathbb{R}^d \text{ compact } : \sup_{t\in\mathbb{R}} \|\chi_{K^c} \mathrm{e}^{-\mathrm{i}tH} f\| \le \varepsilon.$$

$$f \in \mathscr{H}_c \iff \forall K \subset \mathbb{R}^d \text{ compact } : \lim_{T\to\infty} \frac{1}{T}\int_{-T}^{T} \|\chi_K \mathrm{e}^{-\mathrm{i}tH} f\|^2 \,\mathrm{d}t = 0.$$

$$f \in \mathscr{H}_{ac} \implies \forall K \subset \mathbb{R}^d \text{ compact } : \lim_{|t|\to\infty} \|\chi_K \mathrm{e}^{-\mathrm{i}tH} f\| = 0$$

Very general (any self-adjoint op, any Hilbert space).[2]

Refinements ?

2. Ref : e.g. G. Teschl , *Mathematical Methods in Quantum Mechanics. With Applications to Schrödinger Operators*, 2nd ed, GSM 157, AMS 2014.

Example

$$\sigma(-\Delta) = \sigma_{ac}(-\Delta) = [0, \infty)\,.$$

$$\mathrm{e}^{\mathrm{i}t\Delta}\psi(x) = \Big(\frac{1}{4\pi \mathrm{i} t}\Big)^{d/2} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}(x-y)^2/(4t)}\psi(y)\,\mathrm{d}y$$

$$\|\mathrm{e}^{\mathrm{i}t\Delta}\psi\|_\infty \le t^{-d/2}\|\psi\|_1 \qquad \text{(dispersion)}\,,$$

$$\lim_{t\to\infty} \frac{\|x^m \mathrm{e}^{\mathrm{i}t\Delta}\psi\|^2}{t^{2m}} = 4^m \|D^m\psi\|^2 \qquad \text{(ballistic transport - BMS)}\,.$$

Quantum trees

Quantum graphs

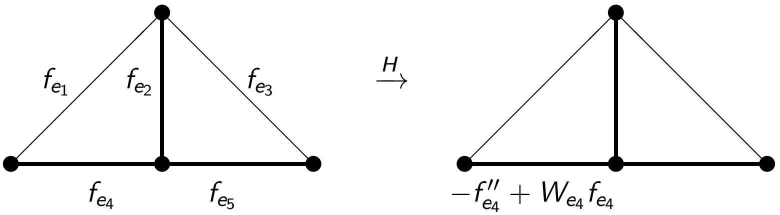

Figure – The Schrödinger operator H acts on each edge by $Hf_e(x) = -f''_e(x) + W_e(x)f_e(x)$. Only one edge is illustrated on the right for clarity. This can be regarded as a singular one-dimensional manifold.

Boundary conditions : e.g. Kirchhoff-type :

$$\begin{cases} f_e(v) = f_{e'}(v) =: f(v) & \text{if } o_e = o_{e'} = v, \\ \sum\limits_{e, o_e = v} f'_e(v) = \alpha_v f(v) & \alpha_v \in \mathbb{R}. \end{cases}$$

Quantum trees

$(q+1)$-regular tree $\mathbb{T}_q$: infinite connected graph with no cycles, each vertex has $(q+1)$ neighbors.

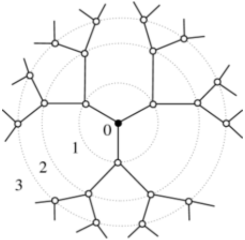

Figure – part of the 3-regular tree $\mathbb{T}_2$. (wikimedia commons).

Identify each edge to $[0, L]$, endow each vertex the same α, each edge the same W.

Spectrum of regular quantum trees

Carlson 97 :

$$\sigma(H) = (\cup_n I_n) \sqcup (\cup_n \delta_n)\,,$$

the spectrum is purely ac in the bands I_n.

The δ_n are eigenvalues of infinite multiplicity, $\delta_n = (\frac{n\pi}{L})^2$ if $W = 0$.

If moreover $\alpha = 0$ then $I_n = [(\frac{(n-1)\pi+\theta}{L})^2, (\frac{n\pi-\theta}{L})^2]$, with $\theta = \arccos \frac{2\sqrt{q}}{q+1}$.

Dispersion on regular quantum trees

Theorem (Ammari, S. - AMP 22)

Let $I_n = [a_n, b_n]$ the n-th AC band. Then as $t \to \infty$, we have uniformly

$$(\mathrm{e}^{\mathrm{i}tH}\mathbf{1}_{ac}(H))(x,y) = \frac{1}{t^{3/2}} \sum_{n \geq 1} f_q(I_n, x, y) + O(t^{-2})$$

for some explicit f_q. In particular, we have the sharp estimate

$$\|\mathrm{e}^{\mathrm{i}tH}\mathbf{1}_{ac}(H)\psi\|_{L^\infty(\mathbf{T}_q)} \leq \frac{C_q}{t^{3/2}}\|\psi\|_{L^1(\mathbf{T}_q)} .$$

If $H\psi = \lambda\psi \implies \mathrm{e}^{-\mathrm{i}tH}\psi = \mathrm{e}^{-\mathrm{i}t\lambda}\psi$
$\implies |(\mathrm{e}^{-\mathrm{i}tH})\psi(x)| = |\psi(x)|$ indep. of time.

Earlier dispersion results

$q = 1, \alpha = 0 \implies$ periodic Schrödinger operator on $\mathbb{R}$.
Here W is the L-periodic potential. Then $\sigma(H) = \sigma_{ac}(H)$.

Firsova 96, Cai 06, Cuccagna 08 : 1d periodic :
$\|\mathrm{e}^{-\mathrm{i}tH}\|_{L^1(\mathbb{R})\to L^\infty(\mathbb{R})} \le t^{-1/3}$ (slower than $t^{-1/2}$, sharp in general).

Ali Mehmeti-Ammari-Nicaise 17 : tadpole, $t^{-1/2}$.

Ali Mehmeti-Ammari-Nicaise 15 : star graph with infinite edges + decaying edge potentials, $t^{-1/2}$.

Banica-Ignat 14 : Finite tree with infinite leads + couplings, $t^{-1/2}$.

Universal covers

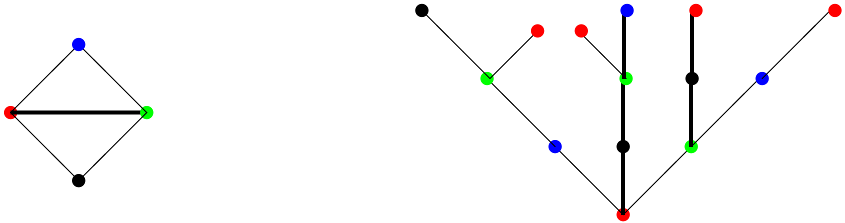

Figure – A simple graph (left) and part of its universal cover (an infinite tree).

We endow the tree with the natural lifted structure $L(e) = L(\pi e)$, $\alpha_v = \alpha_{\pi v}$, $W_e = W_{\pi e}$.

Spectra of infinite trees

Far-reaching extension of Carlson :

Theorem (A.,I.,S.,W. - CMP 21)

If $\mathcal{T}$ is the universal cover of a compact graph of degree ≥ 2 which is not a cycle, then $\sigma(H_{\mathcal{T}})$ consists of bands of pure AC spectrum, plus (possibly) some disjoint eigenvalues.
The Green's functions exist on the AC spectrum and have strictly positive imaginary part.

Note : cycles imply $\mathcal{T} = \mathbb{R}$ with periodic potential $\rightsquigarrow$ already known.

Spectra of infinite trees

Theorem (A.,I.,S.,W. - CMP 21 - Anderson delocalization)

Suppose we weakly perturb the edge lengths and coupling constants of the universal cover, in an i.i.d. fashion. Then the pure AC spectrum remains stable, and we have a strong control over the Green's function.

Note : earlier stability result by Aizenman-Sims-Warzel 06 for $\mathcal{T} = \mathcal{T}_q$ - does not ensure purity, no resolvent control.

Open problems

- Dispersion for periodic Schrödinger operators in $\mathbb{R}^d$, $d \geq 2$.
  $\|\mathrm{e}^{-\mathrm{i}tH}\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \lesssim t^{-d/2}$?
- Dispersion for graphs which are asymptotically regular trees ?
  (Spectral theory by Colin de Verdière and Truc for the discrete case).
  Still $t^{-3/2}$?
  Metric case ?
- Dispersion for universal covers of finite graphs ?
- Ballistic bounds for quantum trees ? (there are some results for combinatorial trees).
- AC spectrum for universal covers of compact manifolds of (variable) negative curvature ?

Some proof ideas

Simpler models

For $H = \mathcal{A} + V$ on a **discrete** tree $\mathbb{T}$, the Green's function obeys recursive formulas :

$$G^{(v|w)}(v,v;z) = \frac{1}{V(v) - z - \sum_{u \in \mathcal{N}_v \setminus \{w\}} G^{(u|v)}(u,u;z)}\,.$$

If $\mathbb{T} = \mathbb{T}_q$ is the $(q+1)$-regular tree and $V = 0$, we get that $\vartheta_z = G^{(v|w)}(v,v;z)$ must obey

$$q\vartheta^2 + z\vartheta + 1 = 0\,.$$

so $\vartheta_z = \frac{-z \pm \sqrt{z^2 - 4q}}{2q}$ is bounded as $\operatorname{Im} z \downarrow 0$, the spectrum is ac and supported on $[-2\sqrt{q}, 2\sqrt{q}]$.

Simpler models

For $\mathcal{A}$ on the universal cover of a finite graph, the $G^{(v|w)}(v, v; z)$ satisfy a finite system of equations. So the $\vartheta_1^z, \ldots, \vartheta_p^z$ are algebraic in z.

$\rightsquigarrow$ finitely many poles on $\mathbb{R}$.

Periodic geometry $\rightsquigarrow$ if one Im $G(v, v; \lambda + \mathrm{i}0)$ vanishes, then all Im $G(w, w; \lambda + \mathrm{i}0)$ vanish.

Anderson model. For $H = \mathcal{A} + \epsilon V_\omega$ on $\mathbb{T}_q$, some averaged Green's function satisfies a fixed point equation with nice properties. So by the IFT, ϑ_λ has a continuous extension in disorder parameter on average.

$\rightsquigarrow$ pure ac spectrum a.s. via Fatou.

Big open problem on $\mathbb{Z}^d$!

Dispersion - simpler discrete case

For $\mathcal{A}$ on $\ell^2(\mathbb{T}_q)$. As the spectrum is AC we have

$$\mathrm{e}^{\mathrm{i}t\mathcal{A}}(v,w) = \frac{1}{\pi}\int_{-2\sqrt{q}}^{2\sqrt{q}} \mathrm{e}^{\mathrm{i}t\lambda}\,\mathrm{Im}\, G^{\lambda+\mathrm{i}0}(v,w)\,\mathrm{d}\lambda\,.$$

Step 1 : let $w = v$ and $\Psi(\lambda) = \mathrm{Im}\, G^{\lambda+\mathrm{i}0}(v,v)$. We have $\Psi(\pm 2\sqrt{q}) = 0$.

$\rightsquigarrow$ Integrating by parts $\rightsquigarrow$ $\mathrm{e}^{\mathrm{i}t\mathcal{A}}(v,v) = \frac{\mathrm{i}}{\pi t}\int_{-2\sqrt{q}}^{2\sqrt{q}} \mathrm{e}^{\mathrm{i}t\lambda}\Psi'(\lambda)\,\mathrm{d}\lambda$.

$\Psi'(\lambda)$ diverges like $(4q-\lambda^2)^{-1/2}$ $\rightsquigarrow$ no more integration by parts.

modulus bound $\rightsquigarrow$ t^{-1}.

Dispersion - simpler discrete case

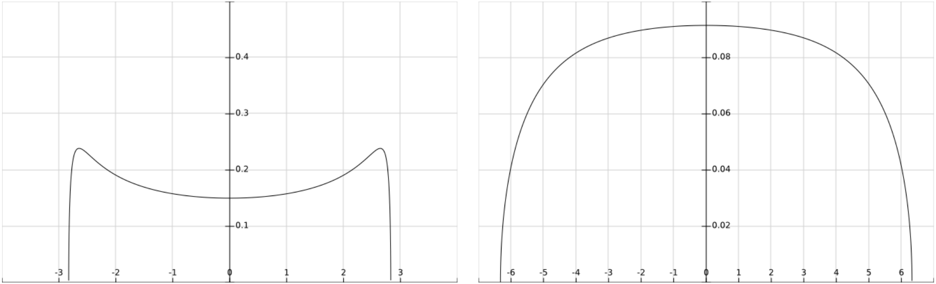

Figure – Graph of $\Psi(\lambda) = \operatorname{Im} G^{\lambda+\mathrm{i}0}(v,v)$ for $q = 2$ (left) and $q = 10$ (right).

Dispersion - simpler discrete case

Step 2 : Get rid of singularity : change of variables $\lambda = 2\sqrt{q}\cos\theta$

$\rightsquigarrow \int_0^\pi \mathrm{e}^{2\mathrm{i}t\sqrt{q}\cos\theta} g(\theta)\,\mathrm{d}\theta,\ \ g$ regular.

Step 3 : Stationary phase method. Phase function : $\phi(\theta) = 2\sqrt{q}\cos\theta$.

Critical points : $\phi'(x) = 0$ at $0, \pi$, $\phi''(x) \neq 0$

$\rightsquigarrow t^{-1/2}$ decay. [3]

Step 4 : Control the remainder.

Step 5 : $w \neq v$. Use spherical functions of $\mathbb{T}_q$, some Chebyshev polynomials properties, get uniform control over $d(v, w)$.

3. Ref : e.g. E. Stein : *Harmonic Analysis : Real-Variable Methods, Orthogonality and Oscillatory Integrals*, PUP 1993.

Dispersion

For H over $L^2(\mathbf{T}_q) \rightsquigarrow$ main target $\rightsquigarrow$ remarkably harder, partly because of the infinitely many bands.

Step 1 : some preliminary calculations for the spectral density $\operatorname{Im} G_{\mathbf{T}_q}^{\lambda+\mathrm{i}0}(o,o)$.

Step 2 : consider $\mathrm{e}^{\mathrm{i}tH}\mathbf{1}_{ac}(H)(o,o)$. Integrating by parts gives

$$\mathrm{e}^{\mathrm{i}tH}\mathbf{1}_{ac}(H)(o,o) = \frac{\mathrm{i}}{\pi t}\int_{\sigma_{ac}(H)} \mathrm{e}^{\mathrm{i}t\lambda}\Psi_1'(\lambda)\,\mathrm{d}\lambda = \frac{\mathrm{i}}{\pi t}\sum_{n\geq 1}\int_{I_n} \mathrm{e}^{\mathrm{i}t\lambda}\Psi_1'(\lambda)\,\mathrm{d}\lambda\,.$$

Proof ideas - Graphs of Im $G_{\mathbf{T}_q}^{\lambda+\mathrm{i}0}(x,x)$

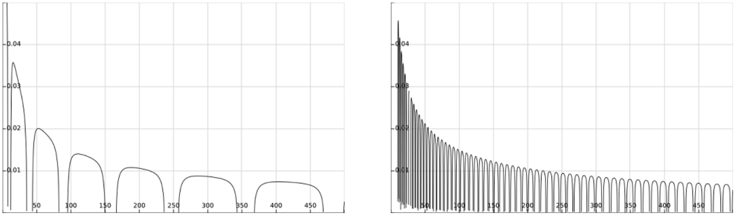

Figure – Spectral density of $-\Delta_{\mathbf{T}_q}$ for $q=2$, $L=1$ (left), $L=10$ (right).

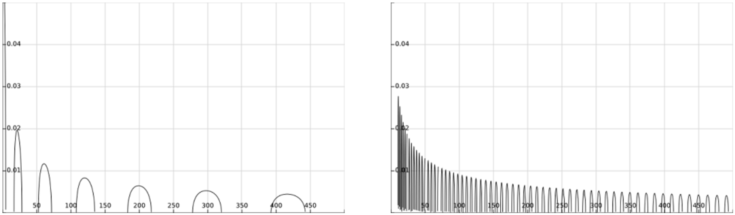

Figure – Spectral density of $-\Delta_{\mathbf{T}_q}$ for $q=10$, $L=1$ (left), $L=10$ (right).

Dispersion

Step 3 : there is again a "natural" change of variables $\lambda = \phi_n(\theta) := w_n^{-1}(2\sqrt{q}\cos\theta)$. This yields

$$\mathrm{e}^{\mathrm{i}tH}\mathbf{1}_{ac}(H)(o,o) = \frac{\mathrm{i}}{\pi t}\sum_{n\geq 1}\int_0^\pi \mathrm{e}^{\mathrm{i}t\phi_n(\theta)} g_n(\theta)\,\mathrm{d}\theta\,.$$

Step 4 : Critical points are $0, \pi$, $\phi_n''(x) \neq 0$ there.

$\rightsquigarrow$ stationary phase method. But $\sum_{n\geq 1} O_n(t^{-2})$ converges ? ?

Dispersion

Some explicit error terms in Stein's and Zworski's books,
but here ϕ_n, g_n quite ugly $\rightsquigarrow$ we want as few derivatives as possible
$\rightsquigarrow$ paper by Olver (74) instead.

Step 5 : Control the series of explicit errors : high frequency solutions of $Hf = \lambda f$ are $\approx \sin, \cos$.
Lots of Taylor-Lagrange.

Step 6 : General case $\mathrm{e}^{\mathrm{i}tH}\mathbf{1}_{ac}(H)(x,y)$. Some resolvent analysis,
Chebyshev identities. Now uniformity in n and $d(x,y)$!