Singular limit solutions for a 2-dimensional semilinear elliptic system of Liouville type in some general case adding singular sources

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Geometric Interpretation

Let (M^2, g_0) a Riemannian surface endowed with a metric g_0. We try to provide M^2 with a new metric g_w conforme to g_0. Under the conformal transformation of the metric $g_w = e^{2w} g_0$, the new Gauss curvature K_{g_w} and the old one K_{g_0} are linked by the following equation

$$-\Delta_{g_0} w - K_{g_0} = K_{g_w} e^{2w}. \tag{1}$$

Observe that, if we take $M^2 = \mathbb{R}^2$ and g_0 is the flat metric then the previous equation becomes

$$-\Delta w = K_{g_w} e^{2w}.$$

Historique

Liouville (1853):

$$-\Delta u = \rho^2 e^u, \tag{2}$$

he derived a representation formula for all solutions of (2) which are defined in $\mathbb{R}^2$.

Suzuki (1990):

$$\begin{cases} -\Delta u = \rho^2 e^u & \text{in} \quad \Omega \subset \mathbb{R}^2 \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3}$$

where ρ is small parameter tends to 0.

Nagasaki and Suzuki (1992):
have studied the behavior of solutions to the following nonlinear eigenvalue problem. More precisely, they consider the following problem

$$\begin{cases} -\Delta u = \rho^2(e^u + e^{\gamma u}) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4}$$

when ρ tends to 0 and $0 < \gamma < \frac{1}{4}$.

Theorem [Suzuki], [Nagasaki and Suzuki]

Let Ω be a smooth bounded domain in $\mathbb{R}^2$. For u_ρ solution of (3) and (4), denote by $\Sigma_\rho = \rho^2 \int_\Omega e^{u_\rho} dx$. Then there are only three possibilities:

i) The $\{\Sigma_\rho\}$ accumulate to 0. Then $\|u_\rho\|_{L^\infty(\Omega)} \longrightarrow 0$ as $\rho \longrightarrow 0$.

ii) The $\{\Sigma_\rho\}$ accumulate to $+\infty$. Then $u_\rho \longrightarrow +\infty$ as $\rho \longrightarrow 0$.

iii) The $\{\Sigma_\rho\}$ accumulate to $8\pi n$, for some positive integer n. Then the limiting function $u^* = \lim_{\rho \to 0} u_\rho$ has n blow-up points, $\{z_1, ..., z_n\}$, where $u_\rho(z_i) \longrightarrow +\infty$ as $\rho \to 0$. Moreover, $(z_1, ..., z_n)$ is a critical point of E.

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where

$$E : (z_1, \cdots, z_n) \in (\mathbb{R}^2)^n \longmapsto \sum_{j=1}^{n} H(z_j, z_j) + \sum_{j \neq l} G(z_j, z_l),$$

$G(z, z')$ is the Green's function defined on $\Omega \times \Omega$, to be solution of

$$\begin{cases} -\Delta G(z, z') = 8\,\pi\,\delta_{z=z'} & \text{in} \quad \Omega \\ G(z, z') = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{5}$$

and

$$H(z, z') = G(z, z') + 4 \log |z - z'|,$$

its regular part.

The limit function u^* is a solution of problem

$$\begin{cases} -\Delta u^* = \sum_{j=1}^{n} 8\,\pi\,\delta_{z_j} & \text{in} \quad \Omega \\ u^* = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{6}$$

Baraket & Pacard (1998): $f(u) = e^u$.

Baraket & Ye (2001): $f(u) = e^u + e^{\gamma u}$, $0 < \gamma < 1$.

Theorem [Baraket et al.]

Let $\Omega \subset \mathbb{R}^2$ and $z_1, ..., z_n \in \Omega$ be given points. Assume that $(z_1, ..., z_n)$ is a nondegenerate critical point of functional E, then there exist $\rho_0 > 0$ and $(u_\rho)_{\rho \leq \rho_0}$ a one parameter family of solutions of (3) and (4) such that

$$\lim_{\rho \longrightarrow 0} u_\rho = u^*,$$

in $\mathcal{C}^{2,\alpha}_{loc}(\Omega \backslash \{z_1, ..., z_n\})$ where u^* is the solution of (6).

Esposito (2005):

$$\begin{cases} -\Delta u = \rho^2 e^u - 4\pi \sum_{i=1}^{N} \alpha_i \delta_{p_i} & \text{in } \Omega \subset \mathbb{R}^2 \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (7)$$

Using the following transformation

$$v = u + \frac{1}{2} \sum_{i=1}^{N} \alpha_i G(., p_i),$$

we get the following "general model" problem

$$\begin{cases} -\triangle v = \rho^2 V e^v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad (8)$$

where

$$V = \prod_{i=1}^{N} |\cdot - p_i|^{2\alpha_i} f. \quad (9)$$

Here ρ tends to 0, f is a positive function with $f(p_i) > 0$ and α_i are positive non-integer numbers.

Let $S = \{q_1, \cdots, q_m\}$ and $\Lambda = \{p_1, \cdots, p_s\}$, $\forall m \in \mathbb{N}$ and $s \in \{1, \cdots, N\}$.

Theorem (Esposito)

Let $\Omega \subset \mathbb{R}^2$ be a smooth open set and $\Omega' = \Omega \cap \{f > 0\}$. We have the following.

1. *Let $S = \{p_{j_1}, \cdots, p_{j_s}\} \subset \Lambda$, then there exist $\rho_0 > 0$ small and a family $(v_\rho)_{\rho<\rho_0}$ of solutions for the problem (7) such that*

$$\lim_{\rho \longrightarrow 0} v_\rho = \sum_{i=1}^{s} (1+\alpha_{j_i}) G(\cdot, p_{j_i}), \qquad \text{in } C^{2,\beta}_{loc}(\Omega \backslash S).$$

2. *Let $S = \{q_1, \cdots, q_m\} \subset \Omega' \backslash \Lambda$ and $(q_1, \cdots, q_m)$ be a nondegenerate critical point of $\mathcal{F}$, then there exist $\rho_0 > 0$ small and a family $(v_\rho)_{\rho<\rho_0}$ of solutions for the problem (7) such that*

$$\lim_{\rho \longrightarrow 0} v_\rho = \sum_{i=1}^{m} G(\cdot, q_i), \qquad \text{in } C^{2,\beta}_{loc}(\Omega \backslash S).$$

3. *Let $S \cap \Lambda = \{p_{j_1}, \cdots, p_{j_s}\}, S \backslash \Lambda = \{q_1, \cdots, q_m\}$ and $(q_1, \cdots, q_m)$ a nondegenerate critical point of the function $\mathcal{F} + \mathcal{G}(\cdot, p_{j_1}, \cdots, p_{j_s})$, then there exist $\rho_0 > 0$ small and a family $(v_\rho)_{\rho<\rho_0}$ of solutions for the problem (7) such that*

$$\lim_{\rho \longrightarrow 0} v_\rho = \sum_{j=1}^{m} G(\cdot, q_j) + \sum_{i=1}^{s} (1+\alpha_{j_i}) G(\cdot, p_{j_i}), \qquad \text{in } C^{2,\beta}_{loc}(\Omega \backslash S).$$

where

$$\mathcal{F}(z_1, \cdots, z_m) = \sum_{j=1}^{m} H(z_j, z_j) + \sum_{i \neq j} G(z_i, z_j) + 4 \sum_{i=1}^{s} \sum_{j=1}^{m} \alpha_i \log(|z_j - p_i|) + 2 \sum_{j=1}^{m} \log(f(z_j))$$

and

$$\mathcal{G}(z_1, \cdots, z_m, w_1, \cdots, w_s) = \sum_{j=1}^{m} \sum_{i=1}^{s} (1 + \alpha_i) G(z_j, w_i).$$

Chanillo & Kiessling (1995):
established a strict isoperimetric inequality and a Pohozaev-Rellich identity for the system

$$-\Delta u_i = \exp\Big(\sum_{j\in\mathfrak{I}} \gamma^{i,j} u_j\Big), \text{ in } \mathbb{R}^2; \;\; i \in \mathfrak{I} = \{1, ..., N\}.$$

Under the finite mass conditions

$$\int_{\mathbb{R}^2} e^{u_i}\, dz < \infty, \;\; i \in \mathfrak{I}$$

and $\{\gamma^{i,j}\} \equiv \gamma \in GL_N(\mathbb{R})$, satisfying

$$\sum_{j\in\mathfrak{I}} \gamma^{i,j} = 1, \;\; i \in \mathfrak{I}.$$

They prove that all solutions $u_i \equiv u$ are radial symmetric and decreasing about some point.

Systems of above equations, find their applications in the physics of charged particle beams. However, we find it interesting that, like Liouville's equation, they have an obvious geometrical significance. A solution $N-$tuple $\{u_j\}$ of (1), defines a set of $N-$metrics, all of which are conformally equivalent to the standard one on $\mathbb{R}^2$.
The metrics pertain to curvature functions

$$K_i = \exp(\sum_{j\neq i} \gamma^{i,j} u_j)$$

which are not prescribed but are determined by the solutions u_j.

N. Trabelsi & M. Trabelsi (2015):

$$\begin{cases} -\Delta u_1 = \rho^2 e^{u_1+\gamma_1 u_2} & \text{in} \quad \Omega \\ -\Delta u_2 = \rho^2 e^{u_2+\gamma_2 u_1} & \text{in} \quad \Omega \\ u_1 = u_2 = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{10}$$

where ρ is small parameter and $\gamma_1, \gamma_2 \in (0,1)$.

Theorem [M. Trabelsi and N.Trabelsi]

Let $\Omega \subset \mathbb{R}^2$ and $z_1, ..., z_n \in \Omega$ be given points. Assume that $(z_1, ..., z_n)$ is a nondegenerate critical point of functional E, then there exist $\rho_0 > 0$ and $(u_i^\rho)_{\rho \leq \rho_0}$ a one parameter family of solutions of (10) such that

$$\begin{cases} \lim_{\rho \to 0} u_1^\rho = \sum_{i=1}^{p} G(\cdot, z_i) & \text{in } \mathcal{C}^{2,\alpha}_{loc}(\Omega \backslash \{z_1, ..., z_p\}) \\ \lim_{\rho \to 0} u_2^\rho = \sum_{j=p+1}^{n} G(\cdot, z_j) & \text{in } \mathcal{C}^{2,\alpha}_{loc}(\Omega \backslash \{z_{p+1}, ..., z_n\}). \end{cases}$$

Baraket & Bazarbacha (2017):

$$\begin{cases} -\Delta v_1 = \rho^2 e^{v_1+\gamma_1 v_2} - 4\pi\alpha_1\delta_{p_1} & \text{in } \Omega \\ -\Delta v_2 = \rho^2 e^{v_2+\gamma_2 v_1} - 4\pi\alpha_2\delta_{p_2} & \text{in } \Omega \\ v_1 = v_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{11}$$

where α_i are positive non-integer numbers and $\gamma_1, \gamma_2 \in (0, 1)$. We proved the existence of singular limit solutions (v_1, v_2) which blow-up in disjoints sets as ρ tends to 0.

Baraket, Sâanouni and Trabelsi (2020):

$$\begin{cases} -\Delta v_1 = \rho^2 e^{\gamma v_1+(1-\gamma)v_2} & \text{in } \Omega \\ -\Delta v_2 = \rho^2 e^{\xi v_2+(1-\xi)v_1} & \text{in } \Omega \\ v_1 = v_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{12}$$

where $\gamma, \xi \in (0, 1)$. We proved the existence of singular limit solutions which blow-up on common points as ρ tends to 0.

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where $\gamma, \xi \in (0,1)$. We proved the existence of singular limit solutions which blow-up on common points as ρ tends to 0.

Problem statement

Let Ω be a regular bounded open domain in $\mathbb{R}^2$. We consider the following elliptic problem :

$$\begin{cases} -\Delta v_1 = \rho^2 e^{\gamma v_1 + (1-\gamma) v_2} - \dfrac{4\pi}{\gamma}\,\alpha_1 \delta_{p_1} - 4\pi\,\alpha_2 \delta_{p_2} & \text{in } \Omega \\ -\Delta v_2 = \rho^2 e^{\xi v_2 + (1-\xi) v_1} - \dfrac{4\pi}{\xi}\,\alpha_3 \delta_{p_3} - 4\pi\,\alpha_2 \delta_{p_2} & \text{in } \Omega \\ v_1 = v_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{13}$$

where ρ is small parameter, α_i are positive non-integer numbers. $\{p_1, p_2, p_3\}$ the set of singular sources and $\gamma, \xi \in (0,1)$ such that $\gamma + \xi > 1$. Then

$$\frac{1-\xi}{\gamma} \in (0,1) \quad \text{and} \quad \frac{1-\gamma}{\xi} \in (0,1).$$

Question:

Does there exist a sequence of solutions (v_1^ρ, v_2^ρ) of (13) which converges to some non trivial singular function on some set as the parameter ρ tends to 0 ?

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If we take

$$\begin{cases} u_1 &= v_1 + \dfrac{\alpha_1}{2\gamma} G(\cdot, p_1) + \dfrac{\alpha_2}{2} G(\cdot, p_2) \\ u_2 &= v_2 + \dfrac{\alpha_3}{2\xi} G(\cdot, p_3) + \dfrac{\alpha_2}{2} G(\cdot, p_2), \end{cases}$$

it is clear that (v_1, v_2) solves (13) if and only if (u_1, u_2) solves the following problem

$$\begin{cases} -\Delta u_1 &= \rho^2 V_1 e^{\gamma u_1 + (1-\gamma) u_2} \quad \text{in} \quad \Omega \\ -\Delta u_2 &= \rho^2 V_2 e^{\xi u_2 + (1-\xi) u_1} \quad \text{in} \quad \Omega \\ u_1 &= u_2 = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{14}$$

where

$$V_1 = |\cdot - p_1|^{2\alpha_1} |\cdot - p_2|^{2\alpha_2} f_1 \quad \text{and} \quad V_2 = |\cdot - p_3|^{2\alpha_3} |\cdot - p_2|^{2\alpha_2} f_2. \tag{15}$$

Here, f_1 and f_2 are smooth non negative functions with $f_1(p_i) > 0, i = 1, 2$ and $f_2(p_i) > 0, i = 2, 3.$

S. Baraket, I. Bazarbacha and R. Chetouane, *Singular limit solutions for a 2-dimensional semilinear elliptic system of Liouville type in some general case adding singular sources (Part I), Nonlinear Anal. 196(2020), 1-51.*

1. $z_1 = p_1$, $z_2 = p_2$ and $z_3 = p_3$.
2. $z_1 = p_1$, $z_2 = p_2$ and $z_3 \neq p_3$.
3. $z_1 \neq p_1$, $z_2 = p_2$ and $z_3 \neq p_3$.

Theorem (1)

Let Ω be a regular open subset of $\mathbb{R}^2$, $\Omega' = \Omega \cap \{f_1 > 0\}$ and $\Omega'' = \Omega \cap \{f_2 > 0\}$. Let $z_1, z_2, z_3 \in \Omega$ be given disjoint points. Let H and G be as above. Suppose that (u_1^ρ, u_2^ρ) is a solution of (14). Then we have

1. *If $z_1 = p_1$, $z_2 = p_2$ and $z_3 \neq p_3$, $z_3 \in \Omega''$. Suppose that*

$$\lim_{\rho \longrightarrow 0} u_1^\rho = \frac{1+\alpha_1}{\gamma} G(\cdot, p_1) + (1+\alpha_2) G(\cdot, p_2) \quad in \ \mathcal{C}^{2,\beta}_{loc}(\Omega \backslash \{p_1, p_2\})$$

and

$$\lim_{\rho \longrightarrow 0} u_2^\rho = \frac{1}{\xi} G(\cdot, z_3) + (1+\alpha_2) G(\cdot, p_2) \quad in \ \mathcal{C}^{2,\beta}_{loc}(\Omega \backslash \{p_2, z_3\}).$$

Then z_3 is the critical point of the functional

$$\mathcal{E}_1(z_3) = \mathcal{F}_1(z_3) + \log\left(|z_3 - p_3|^{2\alpha_3} |z_3 - p_2|^{2\alpha_2} f_2(z_3)\right),$$

where

$$\mathcal{F}_1(z_3) := H(z_3, z_3) + (1+\alpha_2) G(z_3, p_2) + \frac{(1+\alpha_1)(1-\xi)}{\gamma} G(z_3, p_1). \qquad (16)$$

Main results

Theorem (1)

Let Ω be a regular open subset of $\mathbb{R}^2$, $\Omega' = \Omega \cap \{f_1 > 0\}$ and $\Omega'' = \Omega \cap \{f_2 > 0\}$. Let $z_1,\ z_2,\ z_3 \in \Omega$ be given disjoint points. Let H and G be as above. Suppose that (u_1^ρ, u_2^ρ) is a solution of (14). Then we have

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2. If $z_1 \neq p_1$, $z_1 \in \Omega'$, $z_2 = p_2$ and $z_3 \neq p_3$, $z_3 \in \Omega''$. Suppose that

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Then (z_1, z_3) is the critical point of the functional

$$\mathcal{E}_2(z_1, z_3) = \mathcal{F}_2(z_1, z_3) + \frac{1-\xi}{\gamma} \log\left(|z_1 - p_1|^{2\alpha_1} |z_1 - p_2|^{2\alpha_2} f_1(z_1)\right)$$
$$+ \frac{1-\gamma}{\xi} \log\left(|z_3 - p_3|^{2\alpha_3} |z_3 - p_2|^{2\alpha_2} f_2(z_3)\right),$$

where

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Theorem (2)

Let Ω be a regular open subset of $\mathbb{R}^2$ and $z_1,\ z_2,\ z_3 \in \Omega$ be given disjoint points.

1. *Let $z_1 = p_1,\ z_2 = p_2$ and $z_3 = p_3$. We suppose that*

$$|p_2 - p_1|^{2\alpha_1} f_1(p_2) e^{\frac{(1+\alpha_1)(\gamma+\xi-1)}{\gamma} G(p_2,p_1)} = |p_2 - p_3|^{2\alpha_3} f_2(p_2) e^{\frac{(1+\alpha_3)(\gamma+\xi-1)}{\xi} G(p_2,p_3)}.$$

Assume also that

$$0 < \gamma < 2(1+\alpha_1)(1-\xi) \quad \textit{and} \quad 0 < \xi < 2(1+\alpha_3)(1-\gamma).$$

Then there exist $\rho_0 > 0$ and $(u_1^\rho, u_2^\rho)_{\rho \le \rho_0}$ a one parameter family of solutions of (14), such that

$$\lim_{\rho \longrightarrow 0} u_1^\rho = \frac{1+\alpha_1}{\gamma} G(\cdot, p_1) + (1+\alpha_2) G(\cdot, p_2) \quad \textit{in} \quad C^{2,\beta}_{loc}(\Omega \setminus \{p_1, p_2\}),$$

$$\lim_{\rho \longrightarrow 0} u_2^\rho = \frac{1+\alpha_3}{\xi} G(\cdot, p_3) + (1+\alpha_2) G(\cdot, p_2) \quad \textit{in} \quad C^{2,\beta}_{loc}(\Omega \setminus \{p_2, p_3\}).$$

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$$|p_2 - p_1|^{2\alpha_1} f_1(p_2) e^{\frac{(1+\alpha_1)(\gamma+\xi-1)}{\gamma} G(p_2,p_1)} = |p_2 - p_3|^{2\alpha_3} f_2(p_2) e^{\frac{(1+\alpha_3)(\gamma+\xi-1)}{\xi} G(p_2,p_3)}.$$

Assume also that

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Then there exist $\rho_0 > 0$ and $(u_1^\rho, u_2^\rho)_{\rho \le \rho_0}$ a one parameter family of solutions of (14), such that

$$\lim_{\rho \longrightarrow 0} u_1^\rho = \frac{1+\alpha_1}{\gamma} G(\cdot, p_1) + (1+\alpha_2) G(\cdot, p_2) \quad \textit{in } \mathcal{C}^{2,\beta}_{loc}(\Omega \backslash \{p_1, p_2\}),$$

$$\lim_{\rho \longrightarrow 0} u_2^\rho = \frac{1+\alpha_3}{\xi} G(\cdot, p_3) + (1+\alpha_2) G(\cdot, p_2) \quad \textit{in } \mathcal{C}^{2,\beta}_{loc}(\Omega \backslash \{p_2, p_3\}).$$

We suppose that

$$0 < \frac{\gamma+\xi-1}{\gamma} - \frac{\alpha_1}{1+\alpha_1} < \min\left\{\frac{1}{2(2+\alpha_1)}, \frac{\gamma}{4(1+\alpha_1)(1-\xi)}\right\},$$
$$0 < \frac{\gamma+\xi-1}{\xi} - \frac{\alpha_3}{1+\alpha_3} < \min\left\{\frac{1}{2(2+\alpha_3)}, \frac{\xi}{4(1+\alpha_3)(1-\gamma)}\right\}. \quad (18)$$

2. Let $z_1 = p_1$, $z_2 = p_2$ and $z_3 \neq p_3$. Assume that (18) is satisfied and we suppose that

$$|p_2 - p_1|^{2\alpha_1} f_1(p_2) e^{\frac{(1+\alpha_1)(\gamma+\xi-1)}{\gamma} G(p_2,p_1)} = |p_2 - p_3|^{2\alpha_3} f_2(p_2) e^{\frac{\gamma+\xi-1}{\xi} G(p_2,z_3)}.$$

Assume that z_3 is a nondegenerate critical point of the functional $\mathcal{F}_1(z_3)$ given by (16) and f_2 satisfies

$$\nabla\left(|\cdot - p_3|^{2\alpha_3} |\cdot - p_2|^{2\alpha_2} f_2\right)(z_3) = 0. \quad (19)$$

Then there exist $\rho_0 > 0$ and $(u_1^\rho, u_2^\rho)_{\rho \le \rho_0}$ a one parameter family of solutions of (14), such that

$$\lim_{\rho\longrightarrow 0} u_1^\rho = \frac{1+\alpha_1}{\gamma} G(\cdot, p_1) + (1+\alpha_2) G(\cdot, p_2) \quad \text{in } C^{2,\beta}_{loc}(\Omega\backslash\{p_1, p_2\}),$$

$$\lim_{\rho\longrightarrow 0} u_2^\rho = \frac{1}{\xi} G(\cdot, z_3) + (1+\alpha_2) G(\cdot, p_2) \quad \text{in } C^{2,\beta}_{loc}(\Omega\backslash\{p_2, z_3\}).$$

We suppose that

$$0 < \frac{\gamma+\xi-1}{\gamma} - \frac{\alpha_1}{1+\alpha_1} < \min\left\{\frac{1}{2(2+\alpha_1)}, \frac{\gamma}{4(1+\alpha_1)(1-\xi)}\right\},$$
$$0 < \frac{\gamma+\xi-1}{\xi} - \frac{\alpha_3}{1+\alpha_3} < \min\left\{\frac{1}{2(2+\alpha_3)}, \frac{\xi}{4(1+\alpha_3)(1-\gamma)}\right\}. \quad (18)$$

2. Let $z_1 = p_1$, $z_2 = p_2$ and $z_3 \neq p_3$. Assume that (18) is satisfied and we suppose that

$$|p_2 - p_1|^{2\alpha_1} f_1(p_2) e^{\frac{(1+\alpha_1)(\gamma+\xi-1)}{\gamma} G(p_2,p_1)} = |p_2 - p_3|^{2\alpha_3} f_2(p_2) e^{\frac{\gamma+\xi-1}{\xi} G(p_2,z_3)}.$$

Assume that z_3 is a nondegenerate critical point of the functional $\mathcal{F}_1(z_3)$ given by (16) and f_2 satisfies

$$\nabla\left(|\cdot - p_3|^{2\alpha_3} |\cdot - p_2|^{2\alpha_2} f_2\right)(z_3) = 0. \quad (19)$$

Then there exist $\rho_0 > 0$ and $(u_1^\rho, u_2^\rho)_{\rho \le \rho_0}$ a one parameter family of solutions of (14), such that

$$\lim_{\rho \longrightarrow 0} u_1^\rho = \frac{1+\alpha_1}{\gamma} G(\cdot, p_1) + (1+\alpha_2) G(\cdot, p_2) \ \text{ in } \ \mathcal{C}^{2,\beta}_{loc}(\Omega \setminus \{p_1, p_2\}),$$

$$\lim_{\rho \longrightarrow 0} u_2^\rho = \frac{1}{\xi} G(\cdot, z_3) + (1+\alpha_2) G(\cdot, p_2) \ \text{ in } \ \mathcal{C}^{2,\beta}_{loc}(\Omega \setminus \{p_2, z_3\}).$$

3. Let $z_1 \neq p_1$, $z_2 = p_2$ and $z_3 \neq p_3$. Assume that (18) is satisfied and we suppose that

$$|p_2 - p_1|^{2\alpha_1} f_1(p_2) e^{\frac{\gamma+\xi-1}{\gamma} G(p_2, z_1)} = |p_2 - p_3|^{2\alpha_3} f_2(p_2) e^{\frac{\gamma+\xi-1}{\xi} G(p_2, z_3)}.$$

Assume that (z_1, z_3) is a nondegenerate critical point of the functional $\mathcal{F}_2(z_1, z_3)$ given by (17) and f_i for $i = 1, 2$, satisfy

$$\nabla\big(|\cdot - p_1|^{2\alpha_1}|\cdot - p_2|^{2\alpha_2} f_1\big)(z_1) = 0, \quad \nabla\big(|\cdot - p_3|^{2\alpha_3}|\cdot - p_2|^{2\alpha_2} f_2\big)(z_3) = 0. \tag{20}$$

Then there exist $\rho_0 > 0$ and $(u_1^\rho, u_2^\rho)_{\rho \leq \rho_0}$ a one parameter family of solutions of (14), such that

$$\lim_{\rho \longrightarrow 0} u_1^\rho = \frac{1}{\gamma} G(\cdot, z_1) + (1 + \alpha_2) G(\cdot, p_2) \quad \text{in } \mathcal{C}^{2,\beta}_{loc}(\Omega \backslash \{z_1, p_2\}),$$

$$\lim_{\rho \longrightarrow 0} u_2^\rho = \frac{1}{\xi} G(\cdot, z_3) + (1 + \alpha_2) G(\cdot, p_2) \quad \text{in } \mathcal{C}^{2,\beta}_{loc}(\Omega \backslash \{p_2, z_3\}).$$

3. Let $z_1 \neq p_1$, $z_2 = p_2$ and $z_3 \neq p_3$. Assume that (18) is satisfied and we suppose that

$$|p_2 - p_1|^{2\alpha_1} f_1(p_2) e^{\frac{\gamma+\xi-1}{\gamma} G(p_2,z_1)} = |p_2 - p_3|^{2\alpha_3} f_2(p_2) e^{\frac{\gamma+\xi-1}{\xi} G(p_2,z_3)}.$$

Assume that (z_1, z_3) is a nondegenerate critical point of the functional $\mathcal{F}_2(z_1, z_3)$ given by (17) and f_i for $i = 1, 2$, satisfy

$$\nabla\big(|\cdot - p_1|^{2\alpha_1}|\cdot - p_2|^{2\alpha_2} f_1\big)(z_1) = 0, \quad \nabla\big(|\cdot - p_3|^{2\alpha_3}|\cdot - p_2|^{2\alpha_2} f_2\big)(z_3) = 0. \tag{20}$$

Then there exist $\rho_0 > 0$ and $(u_1^\rho, u_2^\rho)_{\rho \le \rho_0}$ a one parameter family of solutions of (14), such that

$$\lim_{\rho \longrightarrow 0} u_1^\rho = \frac{1}{\gamma} G(\cdot, z_1) + (1 + \alpha_2) G(\cdot, p_2) \ \text{ in } \ \mathcal{C}^{2,\beta}_{loc}(\Omega \backslash \{z_1, p_2\}),$$

$$\lim_{\rho \longrightarrow 0} u_2^\rho = \frac{1}{\xi} G(\cdot, z_3) + (1 + \alpha_2) G(\cdot, p_2) \ \text{ in } \ \mathcal{C}^{2,\beta}_{loc}(\Omega \backslash \{p_2, z_3\}).$$

S. Baraket, I. Bazarbacha and R. Chetouane, *Singular limit solutions for a 2-dimensional semilinear elliptic system of Liouville type in some general case adding singular sources (Part II), Nonlinear Anal. 196(2020), 1-95.*

1. $z_1 = p_1$, $z_2 \neq p_2$ and $z_3 \neq p_3$.
2. $z_1 = p_1$, $z_2 \neq p_2$ and $z_3 = p_3$.
3. $z_1 \neq p_1$, $z_2 \neq p_2$ and $z_3 \neq p_3$.

Theorem (3)

Let Ω *be a regular open subset of* $\mathbb{R}^2$, $\Omega' = \Omega \cap \{f_1 > 0\}$ *and* $\Omega'' = \Omega \cap \{f_2 > 0\}$. *Let* $z_1,\ z_2,\ z_3 \in \Omega$ *be given disjoint points. Let* H *and* G *be as above. Suppose that* (u_1^ρ, u_2^ρ) *is a solution of* (14). *Then we have*

1. *If* $z_1 = p_1$, $z_2 \neq p_2$, $z_2 \in (\Omega' \cup \Omega'')$ *and* $z_3 \neq p_3$, $z_3 \in \Omega''$. *Suppose that*

$$\lim_{\rho \longrightarrow 0} u_1^\rho = \frac{1+\alpha_1}{\gamma} G(\cdot, p_1) + G(\cdot, z_2) \quad \text{in } \mathcal{C}^{2,\beta}_{loc}(\Omega \backslash \{p_1, z_2\})$$

and

$$\lim_{\rho \longrightarrow 0} u_2^\rho = \frac{1}{\xi} G(\cdot, z_3) + G(\cdot, z_2) \quad \text{in } \mathcal{C}^{2,\beta}_{loc}(\Omega \backslash \{z_2, z_3\}).$$

Then (z_2, z_3) *is the critical point of the functional*

$$\begin{aligned} \mathcal{E}_1(z_2, z_3) = \quad & \mathcal{F}_1(z_2, z_3) + (1-\xi) \log\left(|z_2 - p_1|^{2\alpha_1} |z_2 - p_2|^{2\alpha_2} f_1(z_2)\right) \\ & + (1-\gamma) \log\left(|z_2 - p_3|^{2\alpha_3} |z_2 - p_2|^{2\alpha_2} f_2(z_2)\right) \\ & + \tfrac{1-\gamma}{\xi} \log\left(|z_3 - p_3|^{2\alpha_3} |z_3 - p_2|^{2\alpha_2} f_2(z_3)\right), \end{aligned}$$

where

$$\begin{aligned} \mathcal{F}_1(z_2, z_3) = \quad & (2-\gamma-\xi) H(z_2, z_2) + \tfrac{1-\gamma}{\xi} H(z_3, z_3) + \tfrac{1-\gamma}{\xi} G(z_2, z_3) \\ & + \tfrac{(1+\alpha_1)(1-\xi)}{\gamma} G(z_2, p_1) + \tfrac{(1+\alpha_1)(1-\xi)(1-\gamma)}{\gamma\xi} G(z_3, p_1). \end{aligned} \tag{21}$$

2. If $z_1 = p_1$, $z_2 \neq p_2$, $z_2 \in (\Omega' \cup \Omega'')$ and $z_3 = p_3$. Suppose that

$$\lim_{\rho \longrightarrow 0} u_1^\rho = \frac{1+\alpha_1}{\gamma} G(\cdot, p_1) + G(\cdot, z_2) \quad \text{in } \mathcal{C}^{2,\beta}_{loc}(\Omega \backslash \{p_1, z_2\})$$

and

$$\lim_{\rho \longrightarrow 0} u_2^\rho = \frac{1+\alpha_3}{\xi} G(\cdot, p_3) + G(\cdot, z_2) \quad \text{in } \mathcal{C}^{2,\beta}_{loc}(\Omega \backslash \{z_2, p_3\}).$$

Then z_2 is the critical point of the functional

$$\begin{aligned} \mathcal{E}_2(z_2) = \quad & \mathcal{F}_2(z_2) + \tfrac{1-\xi}{2-\gamma-\xi} \log\left(|z_2 - p_1|^{2\alpha_1} |z_2 - p_2|^{2\alpha_2} f_1(z_2)\right) \\ & + \tfrac{1-\gamma}{2-\gamma-\xi} \log\left(|z_2 - p_3|^{2\alpha_3} |z_2 - p_2|^{2\alpha_2} f_2(z_2)\right), \end{aligned}$$

where

$$\mathcal{F}_2(z_2) = H(z_2, z_2) + \frac{(1+\alpha_1)(1-\xi)}{\gamma(2-\gamma-\xi)} G(z_2, p_1) + \frac{(1+\alpha_3)(1-\gamma)}{\xi(2-\gamma-\xi)} G(z_2, p_3). \tag{22}$$

3. If $z_1 \neq p_1$, $z_1 \in \Omega'$, $z_2 \neq p_2$, $z_2 \in \Omega' \cup \Omega''$ and $z_3 \neq p_3$, $z_3 \in \Omega''$. Suppose that

$$\lim_{\rho \longrightarrow 0} u_1^\rho = \frac{1}{\gamma} G(\cdot, z_1) + G(\cdot, z_2) \ \text{ in } \ \mathcal{C}^{2,\beta}_{loc}(\Omega \backslash \{z_1, z_2\})$$

and

$$\lim_{\rho \longrightarrow 0} u_2^\rho = \frac{1}{\xi} G(\cdot, z_3) + G(\cdot, z_2) \ \text{ in } \ \mathcal{C}^{2,\beta}_{loc}(\Omega \backslash \{z_2, z_3\}).$$

Then (z_1, z_2, z_3) is the critical point of the functional

$$\begin{aligned} \mathcal{E}_3(z_1, z_2, z_3) = \quad & \mathcal{F}_3(z_1, z_2, z_3) + (1-\xi)\log\left(|z_2 - p_1|^{2\alpha_1}|z_2 - p_2|^{2\alpha_2} f_1(z_2)\right) \\ & +(1-\gamma)\log\left(|z_2 - p_3|^{2\alpha_3}|z_2 - p_2|^{2\alpha_2} f_2(z_2)\right) \\ & +\tfrac{1-\xi}{\gamma}\log\left(|z_1 - p_1|^{2\alpha_1}|z_1 - p_2|^{2\alpha_2} f_1(z_1)\right) \\ & +\tfrac{1-\gamma}{\xi}\log\left(|z_3 - p_3|^{2\alpha_3}|z_3 - p_2|^{2\alpha_2} f_2(z_3)\right), \end{aligned}$$

where

$$\begin{aligned} \mathcal{F}_3(z_1, z_2, z_3) = \quad & \tfrac{1-\xi}{\gamma} H(z_1, z_1) + (2-\gamma-\xi) H(z_2, z_2) + \tfrac{1-\gamma}{\xi} H(z_3, z_3) \\ & + \tfrac{1-\gamma}{\xi} G(z_2, z_3) + \tfrac{1-\xi}{\gamma} G(z_2, z_1) + \tfrac{(1-\xi)(1-\gamma)}{\gamma\xi} G(z_3, z_1). \end{aligned} \quad (23)$$

Let φ is a cut-off function in $C_0^\infty(\Omega)$ such that

$$\begin{cases} \varphi \equiv 1 & \text{in } \; B(z_1, r_0) \cup B(z_3, r_0) \\ \varphi \equiv 0 & \text{in } \; \Omega \setminus (B(z_1, 2r_0) \cup B(z_3, 2r_0)), \end{cases}$$

for $r_0 > 0$ with $B(z_i, 2r_0) \subset \Omega$ for $i = 1, 3$ and $B(z_1, 2r_0) \cap B(z_3, 2r_0) = \emptyset$.

We will introduce some assumptions, we suppose that

$$\begin{aligned} 0 < \quad & \frac{\gamma + \xi - 1}{\gamma} - \frac{\alpha_1}{1 + \alpha_1} < \quad \min\left\{\frac{1}{2(2+\alpha_1)}, \frac{\gamma}{4(1+\alpha_1)(1-\xi)}\right\}, \\ 0 < \quad & \frac{\gamma + \xi - 1}{\xi} - \frac{\alpha_3}{1 + \alpha_3} < \quad \min\left\{\frac{1}{2(2+\alpha_3)}, \frac{\xi}{4(1+\alpha_3)(1-\gamma)}\right\}. \end{aligned} \tag{24}$$

Theorem (4)

1. *Let* $z_1 = p_1$, $z_2 \neq p_2$ *and* $z_3 \neq p_3$. *Assume that* (24) *is satisfied. Suppose that* (z_2, z_3) *is a nondegenerate critical point of the functional* $\mathcal{F}_1(z_2, z_3)$ *given by* (21) *and* f_i *for* $i = 1, 2$, *satisfy*

$$\nabla\big(|\cdot - p_1|^{2\alpha_1}|\cdot - p_2|^{2\alpha_2} f_1\big)(z_2) = 0, \quad \nabla\big(|\cdot - p_3|^{2\alpha_3}|\cdot - p_2|^{2\alpha_2} f_2\big)(z_2) = 0, \tag{25}$$

$$\nabla\big(|\cdot - p_3|^{2\alpha_3}|\cdot - p_2|^{2\alpha_2} f_2\big)(z_3) = 0. \tag{26}$$

Then there exist $\rho_0 > 0$ *and* $(u_1^\rho, u_2^\rho)_{\rho \leq \rho_0}$ *a one parameter family of solutions of* (14), *such that*

$$\lim_{\rho \longrightarrow 0} \varphi u_1^\rho = \frac{1+\alpha_1}{\gamma} \varphi G(\cdot, p_1) \quad \text{in } \mathcal{C}^{2,\beta}_{loc}(\Omega \backslash \{p_1\}),$$

$$\lim_{\rho \longrightarrow 0} \varphi u_2^\rho = \frac{\varphi}{\xi} G(\cdot, z_3) \quad \text{in } \mathcal{C}^{2,\beta}_{loc}(\Omega \backslash \{z_3\})$$

and

$$\lim_{\rho \longrightarrow 0} \Big((1-\xi)u_1^\rho + (1-\gamma)u_2^\rho\Big) = \frac{(1+\alpha_1)(1-\xi)}{\gamma} G(\cdot, p_1) + \frac{1-\gamma}{\xi} G(\cdot, z_3)$$

$$+(2-\gamma-\xi)G(\cdot, z_2) \quad \text{in} \quad \mathcal{C}^{2,\beta}_{loc}(\Omega \backslash \{p_1, z_2, z_3\}).$$

2. Let $z_1 = p_1$, $z_2 \neq p_2$ and $z_3 = p_3$. Assume that (24) is satisfied. Suppose that z_2 is a nondegenerate critical point of the functional $\mathcal{F}_2(z_2)$ given by (22) and f_i for $i = 1, 2$, satisfy

$$\nabla\big(|\cdot - p_1|^{2\alpha_1}|\cdot - p_2|^{2\alpha_2} f_1\big)(z_2) = 0, \quad \nabla\big(|\cdot - p_3|^{2\alpha_3}|\cdot - p_2|^{2\alpha_2} f_2\big)(z_2) = 0. \tag{27}$$

Then there exist $\rho_0 > 0$ and $(u_1^\rho, u_2^\rho)_{\rho \le \rho_0}$ a one parameter family of solutions of (14), such that

$$\lim_{\rho \longrightarrow 0} \varphi u_1^\rho = \frac{1+\alpha_1}{\gamma} \varphi G(\cdot, p_1) \quad \text{in } \ \mathcal{C}^{2,\beta}_{loc}(\Omega \backslash \{p_1\}),$$

$$\lim_{\rho \longrightarrow 0} \varphi u_2^\rho = \frac{1+\alpha_3}{\xi} \varphi G(\cdot, p_3) \quad \text{in } \ \mathcal{C}^{2,\beta}_{loc}(\Omega \backslash \{p_3\})$$

and

$$\lim_{\rho \longrightarrow 0} \Big((1-\xi)u_1^\rho + (1-\gamma)u_2^\rho\Big) = \frac{(1+\alpha_1)(1-\xi)}{\gamma} G(\cdot, p_1) + \frac{(1+\alpha_3)(1-\gamma)}{\xi} G(\cdot, p_3)$$

$$+(2-\gamma-\xi)G(\cdot, z_2) \quad \text{in} \quad \mathcal{C}^{2,\beta}_{loc}(\Omega \backslash \{p_1, z_2, p_3\}).$$

3. Let $z_1 \neq p_1$, $z_2 \neq p_2$ and $z_3 \neq p_3$. Suppose that (z_1, z_2, z_3) is a nondegenerate critical point of the functional $\mathcal{F}_3(z_1, z_2, z_3)$ given by (23) and f_i for $i = 1, 2$, satisfy

$$\nabla\big(|\cdot - p_1|^{2\alpha_1}|\cdot - p_2|^{2\alpha_2} f_1\big)(z_2) = 0, \quad \nabla\big(|\cdot - p_3|^{2\alpha_3}|\cdot - p_2|^{2\alpha_2} f_2\big)(z_2) = 0, \tag{28}$$

$$\nabla\big(|\cdot - p_1|^{2\alpha_1}|\cdot - p_2|^{2\alpha_2} f_1\big)(z_1) = 0, \quad \nabla\big(|\cdot - p_3|^{2\alpha_3}|\cdot - p_2|^{2\alpha_2} f_2\big)(z_3) = 0. \tag{29}$$

Then there exist $\rho_0 > 0$ and $(u_1^\rho, u_2^\rho)_{\rho \leq \rho_0}$ a one parameter family of solutions of (14), such that

$$\lim_{\rho \longrightarrow 0} \varphi u_1^\rho = \frac{\varphi}{\gamma} G(\cdot, z_1) \;\text{ in }\; \mathcal{C}^{2,\beta}_{loc}(\Omega \backslash \{z_1\}),$$

$$\lim_{\rho \longrightarrow 0} \varphi u_2^\rho = \frac{\varphi}{\xi} G(\cdot, z_3) \;\text{ in }\; \mathcal{C}^{2,\beta}_{loc}(\Omega \backslash \{z_3\})$$

and

$$\lim_{\rho \longrightarrow 0} \Big((1-\xi)u_1^\rho + (1-\gamma)u_2^\rho\Big) = \frac{1-\xi}{\gamma} G(\cdot, z_1) + \frac{1-\gamma}{\xi} G(\cdot, z_3) + (2-\gamma-\xi) G(\cdot, z_2)$$

$$\text{in }\; \mathcal{C}^{2,\beta}_{loc}(\Omega \backslash \{z_1, z_2, z_3\}).$$

Theorem (5)

We suppose that

$$|z_2 - p_1|^{2\alpha_1} f_1(z_2) = |z_2 - p_3|^{2\alpha_3} f_2(z_2). \quad (30)$$

1. *Let* $z_1 = p_1$, $z_2 \neq p_2$ *and* $z_3 \neq p_3$. *Assume that* (24) *is satisfied. Suppose that* (z_2, z_3) *is a nondegenerate critical point of the functional* $\mathcal{F}_1(z_2, z_3)$ *given by* (21) *such that*

$$\frac{1+\alpha_1}{\gamma} G(z_2, p_1) = \frac{1}{\xi} G(z_2, z_3), \qquad \frac{1+\alpha_1}{\gamma} \nabla G(\cdot, p_1)(z_2) = \frac{1}{\xi} \nabla G(\cdot, z_3)(z_2)$$

and f_i *for* $i = 1, 2$, *satisfy*

$$\nabla\big(|\cdot - p_1|^{2\alpha_1} |\cdot - p_2|^{2\alpha_2} f_1\big)(z_2) = 0, \quad \nabla\big(|\cdot - p_3|^{2\alpha_3} |\cdot - p_2|^{2\alpha_2} f_2\big)(z_2) = 0,$$

$$\nabla\big(|\cdot - p_3|^{2\alpha_3} |\cdot - p_2|^{2\alpha_2} f_2\big)(z_3) = 0.$$

Then there exist $\rho_0 > 0$ *and* $(u_1^\rho, u_2^\rho)_{\rho \leq \rho_0}$ *a one parameter family of solutions of* (14), *such that*

$$\lim_{\rho \longrightarrow 0} u_1^\rho = \frac{1+\alpha_1}{\gamma} G(\cdot, p_1) + G(\cdot, z_2) \quad \text{in } \mathcal{C}^{2,\beta}_{loc}(\Omega \setminus \{p_1, z_2\}),$$

$$\lim_{\rho \longrightarrow 0} u_2^\rho = \frac{1}{\xi} G(\cdot, z_3) + G(\cdot, z_2) \quad \text{in } \mathcal{C}^{2,\beta}_{loc}(\Omega \setminus \{z_2, z_3\}).$$

2. Let $z_1 = p_1$, $z_2 \neq p_2$ and $z_3 = p_3$. Assume that (24) is satisfied. Suppose that z_2 is a nondegenerate critical point of the functional $\mathcal{F}_2(z_2)$ given by (22) such that

$$\frac{1+\alpha_1}{\gamma} G(z_2\,, p_1) = \frac{1+\alpha_3}{\xi} G(z_2\,, p_3), \quad \frac{1+\alpha_1}{\gamma} \nabla G(\cdot\,, p_1)(z_2) = \frac{1+\alpha_3}{\xi} \nabla G(\cdot\,, p_3)(z_2)$$

and f_i for $i = 1, 2$, satisfy

$$\nabla\big(|\cdot - p_1|^{2\alpha_1} |\cdot - p_2|^{2\alpha_2} f_1\big)\big(z_2\big) = 0, \quad \nabla\big(|\cdot - p_3|^{2\alpha_3} |\cdot - p_2|^{2\alpha_2} f_2\big)\big(z_2\big) = 0.$$

Then there exist $\rho_0 > 0$ and $(u_1^\rho, u_2^\rho)_{\rho \leq \rho_0}$ a one parameter family of solutions of (14), such that

$$\lim_{\rho \longrightarrow 0} u_1^\rho = \frac{1+\alpha_1}{\gamma} G(\cdot, p_1) + G(\cdot, z_2) \ \text{ in } \ \mathcal{C}^{2,\beta}_{loc}(\Omega \backslash \{p_1, z_2\}),$$

$$\lim_{\rho \longrightarrow 0} u_2^\rho = \frac{1+\alpha_3}{\xi} G(\cdot, p_3) + G(\cdot, z_2) \ \text{ in } \ \mathcal{C}^{2,\beta}_{loc}(\Omega \backslash \{z_2, p_3\}).$$

3. Let $z_1 \neq p_1$, $z_2 \neq p_2$ and $z_3 \neq p_3$. Suppose that (z_1, z_2, z_3) is a nondegenerate critical point of the functional $\mathcal{F}_3(z_1, z_2, z_3)$ given by (23) such that

$$\frac{1}{\gamma} G(z_2\,, z_1) = \frac{1}{\xi} G(z_2\,, z_3), \quad \frac{1}{\gamma} \nabla G(\cdot\,, z_1)(z_2) = \frac{1}{\xi} \nabla G(\cdot\,, z_3)(z_2)$$

and f_i for $i = 1, 2$, satisfy

$$\nabla\big(|\cdot - p_1|^{2\alpha_1} |\cdot - p_2|^{2\alpha_2} f_1\big)(z_2) = 0, \quad \nabla\big(|\cdot - p_3|^{2\alpha_3} |\cdot - p_2|^{2\alpha_2} f_2\big)(z_2) = 0,$$

$$\nabla\big(|\cdot - p_1|^{2\alpha_1} |\cdot - p_2|^{2\alpha_2} f_1\big)(z_1) = 0, \quad \nabla\big(|\cdot - p_3|^{2\alpha_3} |\cdot - p_2|^{2\alpha_2} f_2\big)(z_3) = 0.$$

Then there exist $\rho_0 > 0$ and $(u_1^\rho, u_2^\rho)_{\rho \leq \rho_0}$ a one parameter family of solutions of (14), such that

$$\lim_{\rho \longrightarrow 0} u_1^\rho = \frac{1}{\gamma} G(\cdot, z_1) + G(\cdot, z_2) \ \text{ in } \ \mathcal{C}^{2,\beta}_{loc}(\Omega \backslash \{z_1, z_2\}),$$

$$\lim_{\rho \longrightarrow 0} u_2^\rho = \frac{1}{\xi} G(\cdot, z_3) + G(\cdot, z_2) \ \text{ in } \ \mathcal{C}^{2,\beta}_{loc}(\Omega \backslash \{z_2, z_3\}).$$

Hölder Weighted space

We set $\bar{B}_1^* = \bar{B}_1 - \{0\}$.

Definition

Given $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\mu \in \mathbb{R}$, we introduce the Hölder weighted space $\mathcal{C}_\mu^{k,\beta}(\mathbb{R}^2)$ as the space of functions $w \in \mathcal{C}_{loc}^{k,\beta}(\mathbb{R}^2)$ for which the following norm

$$\|w\|_{\mathcal{C}_\mu^{k,\beta}(\mathbb{R}^2)} := \|w\|_{\mathcal{C}^{k,\beta}(\bar{B}_1)} + \sup_{r \geq 1} \left((1+r^2)^{-\mu/2} \|w(r\,\cdot)\|_{\mathcal{C}^{k,\beta}(\bar{B}_1 - B_{1/2})} \right)$$

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Hölder Weighted space

We set $\bar{B}_1^* = \bar{B}_1 - \{0\}$.

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Linearized operators

We define, for $\tau > 0$ the function

$$u_{\varepsilon,\alpha,\tau}(z) := 2\log(1+\alpha) + 2(1+\alpha)\log\tau + 2\log(1+\varepsilon^2) - 2\log(\varepsilon^2 + |\tau z|^{2(1+\alpha)}), \quad (31)$$

solution of

$$-\Delta u = \rho^2 |z|^{2\alpha} e^u \quad \text{in} \quad \mathbb{R}^2 \quad (32)$$

and we denote by $u_{\varepsilon,\tau} = u_{\varepsilon,\tau,0}$, solution of

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We define the associated linear operator of $\Delta u + \rho^2 |z|^{2\alpha} e^u = 0$ about the solution $u_{\varepsilon,\tau,\alpha}$ by

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Lemma

- *Any bounded solution of* $\mathbb{L}\, w = 0$ *defined in* $\mathbb{R}^2$ *is a linear combination of* $\phi_0(z) := \dfrac{1-|z|^2}{1+|z|^2}$ *and* $\phi_i(z) := \dfrac{2\, z_i}{1+|z|^2}$, *for* $i = 1, 2$.
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Proposition

i) Assume that $\mu > 0$, $\mu \notin \mathbb{N}$ *and* $\alpha \notin \mathbb{N}$, *then*

$$\begin{array}{rccc} L_{\alpha,\tau}: & [\mathcal{C}^{2,\beta}_{\mu}(B_1)]_0 & \longrightarrow & \mathcal{C}^{0,\beta}_{\mu-2}(B_1) \\ & w & \longmapsto & L_{\alpha,\tau}\, w \end{array}$$

is surjective.
We denote by $\mathcal{G}_{\mu,\alpha,\tau}$ *to be a right inverse of* $L_{\alpha,\tau}$.
ii) Assume that $\mu > 1$ *and* $\mu \notin \mathbb{N}$, *then*

$$\begin{array}{rccc} L_{\mu}: & \mathcal{C}^{2,\beta}_{\mu}(\mathbb{R}^2) & \longrightarrow & \mathcal{C}^{0,\beta}_{\mu-2}(\mathbb{R}^2) \\ & w & \longmapsto & L\, w \end{array}$$

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iii) Assume that $\delta > 0$ *and* $\delta \notin \mathbb{N}$ *then*

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Interior harmonic extensions

We study the properties of interior harmonic extensions. Given $\varphi \in \mathcal{C}^{2,\beta}(S^1)$, we define $H^{int} = H^{int}(\varphi\,;\cdot) = H^{int}_{\varphi}$ to be the solution of

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We denote by e_1, e_2 the coordinate functions on S^1.

Lemma

1. *If we assume that*

$$\int_{S^1} \varphi\, dv_{S^1} = 0, \tag{34}$$

then there exists $c > 0$ *such that*

$$\|H^{int}_{\varphi}\|_{\mathcal{C}^{2,\beta}_1(\bar{B}_1^*)} \leq c\, \|\varphi\|_{\mathcal{C}^{2,\beta}(S^1)}$$

2. *If we assume that for all* $\ell = 1, 2$

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which decays at infinity.

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Assume that we have

$$\int_{S^1} \tilde{\varphi}\, dv_{S^1} = 0. \tag{37}$$

Then there exists $c > 0$ such that

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where

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is finite, where

$$\mathbf{z} = \{z_1, z_2, z_3\}.$$

Proposition

Assume that $\nu < 0$ and $\nu \notin \mathbb{N}$ then

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Sketch of the proof of Theorem 2

The case: $z_1 = p_1$, $z_2 = p_2$ and $z_3 \neq p_3$

- Rotationally symetric solutions
- The nonlinear interior problem
- The nonlinear exterior Problem
- The nonlinear Cauchy-data matching

Rotationally symmetric solutions

We first describe the rotationally symmetric approximate solutions in $\mathbb{R}^2$ of

$$\begin{cases} -\Delta u_1 = \rho^2|z-p_1|^{2\alpha_1}|z-p_2|^{2\alpha_2} f_1(z) e^{\gamma u_1+(1-\gamma)u_2} & \text{in} \quad \Omega \\ -\Delta u_2 = \rho^2|z-p_3|^{2\alpha_3}|z-p_2|^{2\alpha_2} f_2(z) e^{\xi u_2+(1-\xi)u_1} & \text{in} \quad \Omega, \end{cases} \tag{38}$$

where

$$\rho^2 = \frac{8\,\varepsilon^2}{(1+\varepsilon^2)^2}.$$

For $\bar\alpha = \min(\alpha_1, \alpha_2, \alpha_3)$, $\forall \delta \in (1/2\,,\,1)$ and $\mu \in (1/(2+\bar\alpha), \delta)$, we define

$$\begin{cases} r_{\varepsilon,\alpha} := \max\left(\varepsilon^{2\left(\frac{\gamma+\xi-1}{\gamma}-\frac{\alpha_1}{1+\alpha_1}\right)}, \varepsilon^{2\left(\frac{\gamma+\xi-1}{\xi}-\frac{\alpha_3}{1+\alpha_3}\right)}, \varepsilon^{\frac{1-\mu}{1+\bar\alpha}}, \varepsilon^{\frac{\delta-\mu}{1+\bar\alpha}}\right), \\ r_\varepsilon := \max\left(\varepsilon^{\frac{1}{2}},\, \varepsilon^{\frac{\gamma+\xi-1}{\gamma}},\, \varepsilon^{\frac{\gamma+\xi-1}{\xi}}\right) \end{cases}$$

We define an ansatz for solution of (38), let $\tau_i \geq 0$, for $i = 1, 2, 3$, then we define

$$\widetilde{u}_1(z) = \begin{cases} \frac{1}{\gamma} u_{\varepsilon,\tau_1,\alpha_1}(z - p_1) - \frac{1-\gamma}{\gamma\xi} G(z, z_3) & \\ \quad - \frac{(1+\alpha_2)(1-\gamma)}{\gamma} G(z, p_2) - \frac{1}{\gamma} \log(\gamma |p_1 - p_2|^{2\alpha_2} f_1(p_1)) & z \in B(p_1, r_{\varepsilon,\alpha}) \\ u_{\varepsilon,\tau_2,\alpha_2}(z - p_2) - \frac{\xi}{\gamma+\xi-1} \log(|p_2 - p_1|^{2\alpha_1} f_1(p_2)) & \\ \quad + \frac{1-\gamma}{\gamma+\xi-1} \log(|p_2 - p_3|^{2\alpha_3} f_2(p_2)) & z \in B(p_2, r_{\varepsilon,\alpha}) \\ \frac{1+\alpha_1}{\gamma} G(z, p_1) + (1 + \alpha_2) G(z, p_2) & z \in \Omega \backslash \cup_{i=1,2} B(p_i, r_{\varepsilon,\alpha}) \end{cases}$$

$$\widetilde{u}_2(z) = \begin{cases} \frac{1}{\xi} u_{\varepsilon,\tau_3}(z - z_3) - \frac{(1+\alpha_1)(1-\xi)}{\gamma\xi} G(z, p_1) - \frac{(1+\alpha_2)(1-\xi)}{\xi} G(z, p_2) & \\ \quad - \frac{1}{\xi} \log(\xi |z_3 - p_3|^{2\alpha_3} |z_3 - p_2|^{2\alpha_2} f_2(z_3)) & z \in B(z_3, r_\varepsilon) \\ u_{\varepsilon,\tau_2,\alpha_2}(z - p_2) + \frac{1-\xi}{\gamma+\xi-1} \log(|p_2 - p_1|^{2\alpha_1} f_1(p_2)) & \\ \quad - \frac{\gamma}{\gamma+\xi-1} \log(|p_2 - p_3|^{2\alpha_3} f_2(p_2)) & z \in B(p_2, r_{\varepsilon,\alpha}) \\ \frac{1}{\xi} G(z, z_3) + (1 + \alpha_2) G(z, p_2) & z \in \Omega \backslash (B(p_2, r_{\varepsilon,\alpha}) \cup B(z_3, r_\varepsilon)). \end{cases}$$

Then, for $r=|z-p_1|$, we fix $\delta\in\left(0,\min\left\{1,\,2(1+\alpha_1)(\gamma+\xi-1)/\gamma\right\}\right)$ and $\mu\in(0,1)$, we have

$$\left\|\Delta\tilde{u}_1+\rho^2|z-p_1|^{2\alpha_1}|z-p_2|^{2\alpha_2}f_1(z)\;e^{\gamma\tilde{u}_1+(1-\gamma)\tilde{u}_2}\right\|_{\mathcal{C}^{0,\beta}_{\mu-2}(B_{r_{\varepsilon,\alpha}}(p_1))}\leq Cr_{\varepsilon,\alpha}$$

$$\left\|\Delta\tilde{u}_2+\rho^2|z-p_3|^{2\alpha_3}|z-p_2|^{2\alpha_2}f_2(z)\;e^{\xi\tilde{u}_2+(1-\xi)\tilde{u}_1}\right\|_{\mathcal{C}^{0,\beta}_{\delta-2}(B_{r_{\varepsilon,\alpha}}(p_1))}\leq Cr_{\varepsilon,\alpha}.$$

Then, for $r=|z-p_2|$ and $\mu\in(0,1)$, we have

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Then, for $r=|z-z_3|$, we fix $\delta\in(0,\,\frac{\gamma+\xi-1}{\xi})$, $\mu\in(1,2)$ and using the condition (19), we have

$$\left\|\Delta\tilde{u}_1+\rho^2|z-p_1|^{2\alpha_1}|z-p_2|^{2\alpha_2}f_1(z)\;e^{\gamma\tilde{u}_1+(1-\gamma)\tilde{u}_2}\right\|_{\mathcal{C}^{0,\beta}_{\delta-2}(B_{r_\varepsilon}(z_3))}\leq Cr_\varepsilon^2$$

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The nonlinear interior problem

- **In $\mathbf{B(p_1, r_{\varepsilon,\alpha})}$**: We look for a solution of (38) of the form

$$\begin{cases} v_1(z) = & \tilde{u}_1(z) + H^{int}(\varphi_1^1;\ (z-p_1)/r_{\varepsilon,\alpha}) +\ h_1^1(z) \\ v_2(z) = & \tilde{u}_2(z) + H^{int}(\varphi_2^1;\ (z-p_1)/r_{\varepsilon,\alpha}) + h_2^1(z). \end{cases}$$

This amounts to solve the equations

$$\begin{cases} \mathbb{L}_{\alpha,\tau_1} h_1^1 = \frac{8\varepsilon^2 \tau_1^{2(1+\alpha_1)}(1+\alpha_1)^2 |z-p_1|^{2\alpha_1}}{\gamma\left(\varepsilon^2+(\tau_1|z-p_1|)^{2(1+\alpha_1)}\right)^2} \Big[\frac{|z-p_2|^{2\alpha_2} f_1(z)}{|p_1-p_2|^{2\alpha_2} f_1(p_1)} \\ \qquad\qquad \times e^{\gamma(h_1^1+H_1^{int,1})+(1-\gamma)\,(h_2^1+H_2^{int,1})} - \gamma h_1^1 - 1 \Big] \\ -\Delta h_2^1 = \frac{8\ \varepsilon^2 (1+\alpha_1)^{2\frac{1-\xi}{\gamma}} \tau_1^{2\frac{(1+\alpha_1)(1-\xi)}{\gamma}} |z-p_3|^{2\alpha_3} |z-p_2|^{2\alpha_2} f_2(z)}{\gamma^{\frac{1-\xi}{\gamma}} (1+\varepsilon^2)^{2\frac{\gamma+\xi-1}{\gamma}} (\varepsilon^2+(\tau_1|z-p_1|)^{2(1+\alpha_1)})^{2\frac{1-\xi}{\gamma}} |p_1-p_2|^{2\alpha_2\frac{1-\xi}{\gamma}} f_1(p_1)^{\frac{1-\xi}{\gamma}}} \\ \qquad\qquad \times e^{\frac{\gamma+\xi-1}{\gamma\xi} G(z,z_3) + \frac{(1+\alpha_2)(\gamma+\xi-1)}{\gamma} G(z,p_2) + \xi(h_2^1+\ H_2^{int,1}) + (1-\ \xi)(h_1^1+H_1^{int,1})}. \end{cases} \tag{39}$$

We denote by

$$\mathbb{L}_{\alpha,\tau_1} h_1^1 = \mathcal{R}_1(h_1^1, h_2^1) \quad \text{and} \quad -\Delta h_2^1 = \mathcal{R}_2(h_1^1, h_2^1).$$

To find a solution of (39), it is enough to find a fixed point (h_1^1, h_2^1) in a small ball of $\mathcal{C}_\mu^{2,\beta}\left(B_{r_{\varepsilon,\alpha}}(p_1)\right) \times \mathcal{C}_\delta^{2,\beta}\left(B_{r_{\varepsilon,\alpha}}(p_1)\right)$ solutions of

$$\begin{cases} h_1^1 &= \mathcal{G}_{\mu,\alpha,\tau_1} \circ \mathcal{R}_1(h_1^1, h_2^1) = \mathcal{N}_1(h_1^1, h_2^1) \\ h_2^1 &= \mathcal{K}_\delta \circ \mathcal{E}_{r_{\varepsilon,\alpha}} \circ \mathcal{R}_2(h_1^1, h_2^1) = \mathcal{M}_1(h_1^1, h_2^1), \end{cases} \tag{40}$$

where for all $\delta \geq 0 : \xi_\sigma : \mathcal{C}_\mu^{0,\beta}(\bar{B}_\sigma) \longrightarrow \mathcal{C}_\mu^{0,\beta}(\mathbb{R}^2)$ the extension operator.

We summarize this in the following proposition.

Proposition

Given $\kappa > 0$, $\alpha_1 \notin \mathbb{N}$, $\delta \in (0, \delta_1)$ and $\mu \in (0, \delta)$ there exist $\varepsilon_\kappa > 0$ and $c_\kappa > 0$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, there exists a unique (h_1^1, h_2^1) solution of (40) such that

$$\|(h_1^1, h_2^1)\|_{\mathcal{C}_\mu^{2,\beta}(B_{r_{\varepsilon,\alpha}}(p_1)) \times \mathcal{C}_\delta^{2,\beta}(B_{r_{\varepsilon,\alpha}}(p_1))} \leqslant c_\kappa r_{\varepsilon,\alpha}.$$

Hence (v_1, v_2) defined by (39) solves (38) in $B_{r_{\varepsilon,\alpha}}(p_1)$.

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$$\|(h_1^1, h_2^1)\|_{\mathcal{C}_\mu^{2,\beta}\left(B_{r_{\varepsilon,\alpha}}(p_1)\right) \times \mathcal{C}_\delta^{2,\beta}\left(B_{r_{\varepsilon,\alpha}}(p_1)\right)} \leqslant c_\kappa r_{\varepsilon,\alpha}.$$

Hence (v_1, v_2) defined by (39) solves (38) in $B_{r_{\varepsilon,\alpha}}(p_1)$.

We denote by

$$\mathbb{L}_{\alpha,\tau_1} h_1^1 = \mathcal{R}_1(h_1^1, h_2^1) \quad \text{and} \quad -\Delta h_2^1 = \mathcal{R}_2(h_1^1, h_2^1).$$

To find a solution of (39), it is enough to find a fixed point (h_1^1, h_2^1) in a small ball of $\mathcal{C}_\mu^{2,\beta}\left(B_{r_{\varepsilon,\alpha}}(p_1)\right) \times \mathcal{C}_\delta^{2,\beta}\left(B_{r_{\varepsilon,\alpha}}(p_1)\right)$ solutions of

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Given $\kappa > 0$, $\alpha_1 \notin \mathbb{N}$, $\delta \in (0, \delta_1)$ and $\mu \in (0, \delta)$ there exist $\varepsilon_\kappa > 0$ and $c_\kappa > 0$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, there exists a unique (h_1^1, h_2^1) solution of (40) such that

$$\|(h_1^1, h_2^1)\|_{\mathcal{C}_\mu^{2,\beta}\left(B_{r_{\varepsilon,\alpha}}(p_1)\right) \times \mathcal{C}_\delta^{2,\beta}\left(B_{r_{\varepsilon,\alpha}}(p_1)\right)} \leqslant c_\kappa r_{\varepsilon,\alpha}.$$

Hence (v_1, v_2) defined by (39) solves (38) in $B_{r_{\varepsilon,\alpha}}(p_1)$.

• In $\mathbf{B}(\mathbf{p_2}, \mathbf{r}_{\varepsilon,\alpha})$: We look for a solution of (38) of the form

$$\begin{cases} v_1(z) = & \tilde{u}_1(z) + H^{int}(\varphi_1^2;\ (z-p_2)/r_{\varepsilon,\alpha}) + \ h_1^2(z) \\ v_2(z) = & \tilde{u}_2(z) + H^{int}(\varphi_2^2;\ (z-p_2)/r_{\varepsilon,\alpha}) + h_2^2(z). \end{cases}$$

This amounts to solve the equations

$$\begin{cases} \mathbb{L}_{\alpha,\tau_2} h_1^2 = & \frac{8\varepsilon^2\tau_2^{2(1+\alpha_2)}(1+\alpha_2)^2|z-p_2|^{2\alpha_2}}{\left(\varepsilon^2+(\tau_2|z-p_2|)^{2(1+\alpha_2)}\right)^2}\left[\frac{|z-p_1|^{2\alpha_1}f_1(z)}{|p_2-p_1|^{2\alpha_1}f_1(p_2)}\right. \\ & \left.\times e^{\gamma\left(h_1^2+H_1^{int,2}\right)+(1-\gamma)\left(h_2^2+H_2^{int,2}\right)} - h_1^2 - 1\right] \\ \mathbb{L}_{\alpha,\tau_2} h_2^2 = & \frac{8\varepsilon^2\tau_2^{2(1+\alpha_2)}(1+\alpha_2)^2|z-p_2|^{2\alpha_2}}{\left(\varepsilon^2+(\tau_2|z-p_2|)^{2(1+\alpha_2)}\right)^2}\left[\frac{|z-p_3|^{2\alpha_3}f_2(z)}{|p_2-p_3|^{2\alpha_3}f_2(p_2)}\right. \\ & \left.\times e^{\xi\left(h_2^2+H_2^{int,2}\right)+(1-\xi)\left(h_1^2+H_1^{int,2}\right)} - h_2^2 - 1\right]. \end{cases} \tag{41}$$

We denote by

$$\mathbb{L}_{\alpha,\tau_2} h_1^2 = \mathcal{R}_3(h_1^2, h_2^2) \quad \text{and} \quad \mathbb{L}_{\alpha,\tau_2} h_2^2 = \mathcal{R}_4(h_1^2, h_2^2).$$

To find a solution of (41), it is enough to find a fixed point (h_1^2, h_2^2) in a small ball of $\mathcal{C}_\mu^{2,\beta}\left(B_{r_{\varepsilon,\alpha}}(p_2)\right) \times \mathcal{C}_\mu^{2,\beta}\left(B_{r_{\varepsilon,\alpha}}(p_2)\right)$ solutions of

$$\begin{cases} h_1^2 &= \mathcal{G}_{\mu,\alpha,\tau_2} \circ \mathcal{R}_3(h_1^2, h_2^2) = \mathcal{N}_2(h_1^2, h_2^2) \\ h_2^2 &= \mathcal{G}_{\mu,\alpha,\tau_2} \circ \mathcal{R}_4(h_1^2, h_2^2) = \mathcal{M}_2(h_1^2, h_2^2). \end{cases} \tag{42}$$

We summarize this in the following proposition.

Proposition

Given $\kappa > 0$, $\alpha_2 \notin \mathbb{N}$, γ_0, ξ_0 and μ in $(0,1)$ there exist $\varepsilon_\kappa > 0$ and $c_\kappa > 0$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\gamma \in (\gamma_0, 1)$ and $\xi \in (\xi_0, 1)$, there exists a unique (h_1^2, h_2^2) solution of (42) such that

$$\|(h_1^2, h_2^2)\|_{\mathcal{C}_\mu^{2,\beta}(B_{r_{\varepsilon,\alpha}}(p_2)) \times \mathcal{C}_\mu^{2,\beta}(B_{r_{\varepsilon,\alpha}}(p_2))} \leqslant c_\kappa r_{\varepsilon,\alpha}.$$

Hence (v_1, v_2) defined by (41) solves (38) in $B_{r_{\varepsilon,\alpha}}(p_2)$.

We denote by

$$\mathbb{L}_{\alpha,\tau_2} h_1^2 = \mathcal{R}_3(h_1^2, h_2^2) \quad \text{and} \quad \mathbb{L}_{\alpha,\tau_2} h_2^2 = \mathcal{R}_4(h_1^2, h_2^2).$$

To find a solution of (41), it is enough to find a fixed point (h_1^2, h_2^2) in a small ball of $\mathcal{C}_\mu^{2,\beta}\left(B_{r_{\varepsilon,\alpha}}(p_2)\right) \times \mathcal{C}_\mu^{2,\beta}\left(B_{r_{\varepsilon,\alpha}}(p_2)\right)$ solutions of

$$\left\{ \begin{array}{lcl} h_1^2 & = & \mathcal{G}_{\mu,\alpha,\tau_2} \circ \mathcal{R}_3(h_1^2, h_2^2) = \mathcal{N}_2(h_1^2, h_2^2) \\ h_2^2 & = & \mathcal{G}_{\mu,\alpha,\tau_2} \circ \mathcal{R}_4(h_1^2, h_2^2) = \mathcal{M}_2(h_1^2, h_2^2). \end{array} \right. \tag{42}$$

We summarize this in the following proposition.

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Given $\kappa > 0$, $\alpha_2 \notin \mathbb{N}$, γ_0, ξ_0 and μ in $(0,1)$ there exist $\varepsilon_\kappa > 0$ and $c_\kappa > 0$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\gamma \in (\gamma_0, 1)$ and $\xi \in (\xi_0, 1)$, there exists a unique (h_1^2, h_2^2) solution of (42) such that

$$\|(h_1^2, h_2^2)\|_{\mathcal{C}_\mu^{2,\beta}\left(B_{r_{\varepsilon,\alpha}}(p_2)\right) \times \mathcal{C}_\mu^{2,\beta}\left(B_{r_{\varepsilon,\alpha}}(p_2)\right)} \leqslant c_\kappa r_{\varepsilon,\alpha}.$$

Hence (v_1, v_2) defined by (41) solves (38) in $B_{r_{\varepsilon,\alpha}}(p_2)$.

• In $\mathbf{B(z_3, r_\varepsilon)}$: Using the following transformation

$$\begin{cases} v_1(z) = u_1(\frac{\varepsilon}{\tau_3}z) \\ v_2(z) = u_2(\frac{\varepsilon}{\tau_3}z) + \frac{4}{\xi}\log\varepsilon - \frac{2}{\xi}\log(\frac{\tau_3(1+\varepsilon^2)}{2}). \end{cases} \quad (43)$$

Then the system (38) can be written as

$$\begin{cases} -\Delta v_1 = 2C_{3,\varepsilon}^{2\frac{\gamma+\xi-1}{\xi}}\varepsilon^{4\frac{\gamma+\xi-1}{\xi}}|\frac{\varepsilon}{\tau_3}z - p_1|^{2\alpha_1}|\frac{\varepsilon}{\tau_3}z - p_2|^{2\alpha_2}f_1(\frac{\varepsilon}{\tau_3}z)e^{\gamma v_1+(1-\gamma)v_2} & \text{in } B_{R_\varepsilon^3}(z_3) \\ -\Delta v_2 = 2|\frac{\varepsilon}{\tau_3}z - p_3|^{2\alpha_3}|\frac{\varepsilon}{\tau_3}z - p_2|^{2\alpha_2}f_2(\frac{\varepsilon}{\tau_3}z)e^{\xi v_2+(1-\xi)v_1} & \text{in } B_{R_\varepsilon^3}(z_3), \end{cases} \quad (44)$$

where $C_{3,\varepsilon} := \dfrac{2}{\tau_3(1+\varepsilon^2)}$ and $R_\varepsilon^3 := \dfrac{\tau_3 r_\varepsilon}{\varepsilon}$.

In $B(z_3, R_\varepsilon^3)$: we look for a solution of (44) of the form

$$\begin{cases} v_1(z) = \frac{1+\alpha_1}{\gamma} G(\frac{\varepsilon}{\tau_3} z, p_1) + (1+\alpha_2) G(\frac{\varepsilon}{\tau_3} z, p_2) + H^{int}_{\varphi_1^3}(\frac{z-z_3}{R_\varepsilon^3}) + h_1^3(z) \\ v_2(z) = \frac{1}{\xi} u_{\varepsilon,\tau_3}(z-z_3) - \frac{(1+\alpha_1)(1-\xi)}{\gamma\xi} G(\frac{\varepsilon}{\tau_3} z, p_1) - \frac{(1+\alpha_2)(1-\xi)}{\xi} G(\frac{\varepsilon}{\tau_3} z, p_2) \\ \qquad -\frac{1}{\xi} \log(\xi |\frac{\varepsilon}{\tau_3} z_3 - p_3|^{2\alpha_3} |\frac{\varepsilon}{\tau_3} z_3 - p_2|^{2\alpha_2} f_2(\frac{\varepsilon}{\tau_3} z_3)) + H^{int}_{\varphi_2^3}(\frac{z-z_3}{R_\varepsilon^3}) + h_2^3(z). \end{cases}$$

This amounts to solve the equations

$$\begin{cases} -\Delta h_1^3 = \dfrac{2 C_{3,\varepsilon}^{2\frac{\gamma+\xi-1}{\xi}} 4^{\frac{1-\gamma}{\xi}} \varepsilon^{4\frac{\gamma+\xi-1}{\xi}} |\frac{\varepsilon}{\tau_3} z - p_1|^{2\alpha_1} |\frac{\varepsilon}{\tau_3} z - p_2|^{2\alpha_2} f_1(\frac{\varepsilon}{\tau_3} z)}{\xi^{\frac{1-\gamma}{\xi}} |\frac{\varepsilon}{\tau_3} z_3 - p_1|^{2\alpha_1 \frac{1-\gamma}{\xi}} |\frac{\varepsilon}{\tau_3} z_3 - p_2|^{2\alpha_2 \frac{1-\gamma}{\xi}} (1+r^2)^{2\frac{1-\gamma}{\xi}} f_2(\frac{\varepsilon}{\tau_3} z_3)^{\frac{1-\gamma}{\xi}}} \\ \qquad \times e^{\frac{(1+\alpha_1)(\gamma+\xi-1)}{\gamma\xi} G(\frac{\varepsilon}{\tau_3} z, p_1) + \frac{(1+\alpha_2)(\gamma+\xi-1)}{\xi} G(\frac{\varepsilon}{\tau_3} z, p_2) + \gamma(h_1^3 + H_1^{int,3}) + (1-\gamma)(h_2^3 + H_2^{int,3})} \\ \mathbb{L} h_2^3 = \frac{8}{\xi(1+r^2)^2} \Big[\dfrac{|\frac{\varepsilon}{\tau_3} z - p_3|^{2\alpha_3} |\frac{\varepsilon}{\tau_3} z - p_2|^{2\alpha_2} f_2(\frac{\varepsilon}{\tau_3} z)}{|\frac{\varepsilon}{\tau_3} z_3 - p_3|^{2\alpha_3} |\frac{\varepsilon}{\tau_3} z_3 - p_2|^{2\alpha_2} f_2(\frac{\varepsilon}{\tau_3} z_3)} e^{\xi(h_2^3 + H_2^{int,3}) + (1-\xi)(h_1^3 + H_1^{int,3})} - \xi h_2^3 - 1 \Big]. \end{cases} \tag{45}$$

We denote by

$$-\Delta h_1^3 = \mathcal{R}_5(h_1^3, h_2^3) \quad \text{and} \quad \mathbb{L} h_2^3 = \mathcal{R}_6(h_1^3, h_2^3).$$

To find a solution of (45), it is enough to find a fixed point (h_1^3, h_2^3) in a small ball of $\mathcal{C}_\delta^{2,\beta}(\mathbb{R}^2) \times \mathcal{C}_\mu^{2,\beta}(\mathbb{R}^2)$ solutions of

$$\begin{cases} h_1^3 & = & \mathcal{K}_\delta \circ \mathscr{E}_\delta \circ \mathcal{R}_5(h_1^3, h_2^3) = \mathcal{N}_3(h_1^3, h_2^3) \\ h_2^3 & = & \mathcal{G}_\mu \circ \mathscr{E}_\mu \circ \mathcal{R}_6(h_1^3, h_2^3) = \mathcal{M}_3(h_1^3, h_2^3), \end{cases} \tag{46}$$

where for all $\delta \geq 0$: $\xi_\sigma : \mathcal{C}_\mu^{0,\beta}(\bar{B}_\sigma) \longrightarrow \mathcal{C}_\mu^{0,\beta}(\mathbb{R}^2)$ the extension operator.

We summarize this in the following proposition.

Proposition

Given $\kappa > 0$, $\mu \in (1,2)$, $\delta \in \left(0, \frac{\gamma+\xi-1}{\xi}\right)$ and assume that the condition (19) is satisfied, then there exist $\varepsilon_\kappa > 0$, $c_\kappa > 0$ and $\xi_0 \in (0,1)$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\xi \in (\xi_0, 1)$, there exists a unique (h_1^3, h_2^3) solution of (46) such that

$$\|(h_1^3, h_2^3)\|_{\mathcal{C}_\delta^{2,\beta}(\mathbb{R}^2)\times \mathcal{C}_\mu^{2,\beta}(\mathbb{R}^2)} \leq 2c_\kappa r_\varepsilon^2.$$

Hence (v_1, v_2) defined by (45) solves (44) in $B_{R_\varepsilon^3}(z_3)$.

We denote by

$$-\Delta h_1^3 = \mathcal{R}_5(h_1^3, h_2^3) \quad \text{and} \quad \mathbb{L} h_2^3 = \mathcal{R}_6(h_1^3, h_2^3).$$

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$$\begin{cases} h_1^3 &= \mathcal{K}_\delta \circ \mathscr{E}_\delta \circ \mathcal{R}_5(h_1^3, h_2^3) = \mathcal{N}_3(h_1^3, h_2^3) \\ h_2^3 &= \mathcal{G}_\mu \circ \mathscr{E}_\mu \circ \mathcal{R}_6(h_1^3, h_2^3) = \mathcal{M}_3(h_1^3, h_2^3), \end{cases} \tag{46}$$

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$$\|(h_1^3, h_2^3)\|_{C_\delta^{2,\beta}(\mathbb{R}^2) \times C_\mu^{2,\beta}(\mathbb{R}^2)} \leq 2 c_\kappa r_\varepsilon^2.$$

Hence (v_1, v_2) defined by (45) solves (44) in $B_{R_\varepsilon^3}(z_3)$.

The nonlinear exterior Problem

Given $\tilde{\mathbf{q}} := (p_1, p_2, \tilde{z}_3) \in \Omega^3$ close to $\mathbf{q} := (p_1, p_2, z_3)$, $\lambda := (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$ close to 0, such that

$$|\lambda_i| \leqslant r^2_{\varepsilon,\alpha} \quad \text{and} \quad |z_3 - \tilde{z}_3| \leqslant \kappa r_\varepsilon. \tag{47}$$

We define

$$\begin{cases} \tilde{\mathbf{w}}_1 := \dfrac{1+\lambda_1+\alpha_1}{\gamma} G(\cdot, p_1) + (1+\lambda_2+\alpha_2) G(\cdot, p_2) + \displaystyle\sum_{i=1}^{2} \chi_{r_0}(\cdot - p_i) H^{ext}(\tilde{\varphi}_1^i; (\cdot - p_i)/r_{\varepsilon,\alpha}) \\ \qquad +\chi_{r_0}(\cdot - \tilde{z}_3) H^{ext}(\tilde{\varphi}_1^3; (\cdot - \tilde{z}_3)/r_\varepsilon) \\ \tilde{\mathbf{w}}_2 := \dfrac{1+\lambda_3}{\xi} G(\cdot, \tilde{z}_3) + (1+\lambda_2+\alpha_2) G(\cdot, p_2) + \displaystyle\sum_{i=1}^{2} \chi_{r_0}(\cdot - p_i) H^{ext}(\tilde{\varphi}_2^i; (\cdot - p_i)/r_{\varepsilon,\alpha}) \\ \qquad +\chi_{r_0}(\cdot - \tilde{z}_3) H^{ext}(\tilde{\varphi}_2^3; (\cdot - \tilde{z}_3)/r_\varepsilon). \end{cases}$$

Here χ_{r_0} is a cut-off function identically equal to 1 in $B_{r_0/2}$ and identically equal to 0 outside B_{r_0}.

We would like to find a solution of the system

$$\begin{cases} -\Delta u_1 &= \rho^2|z-p_1|^{2\alpha_1}|z-p_2|^{2\alpha_2} f_1(z) e^{\gamma u_1+(1-\gamma)u_2} \\ -\Delta u_2 &= \rho^2|z-p_3|^{2\alpha_3}|z-p_2|^{2\alpha_2} f_2(z) e^{\xi u_2+(1-\xi)u_1} \end{cases} \tag{48}$$

which is defined in $\bar\Omega - \{B_{r_{\varepsilon,\alpha}}(p_1) \cup B_{r_{\varepsilon,\alpha}}(p_2) \cup B_{r_\varepsilon}(\tilde z_3)\}$, with $u_1 = \tilde{\mathbf{w}}_1 + \tilde v_1$ a perturbation of $\tilde{\mathbf{w}}_1$ and $u_2 = \tilde{\mathbf{w}}_2 + \tilde v_2$ a perturbation of $\tilde{\mathbf{w}}_2$.

This amounts to solve in $\bar\Omega - \{B_{r_{\varepsilon,\alpha}}(p_1) \cup B_{r_{\varepsilon,\alpha}}(p_2) \cup B_{r_\varepsilon}(\tilde z_3)\}$

$$\begin{cases} -\Delta \tilde v_1 &= \rho^2|z-p_1|^{2\alpha_1}|z-p_2|^{2\alpha_2} f_1(z) \left(e^{\gamma(\tilde{\mathbf{w}}_1+\tilde v_1)+(1-\gamma)(\tilde{\mathbf{w}}_2+\tilde v_2)}\right) + \Delta\tilde{\mathbf{w}}_1 \\ -\Delta \tilde v_2 &= \rho^2|z-p_3|^{2\alpha_3}|z-p_2|^{2\alpha_2} f_2(z) \left(e^{\xi(\tilde{\mathbf{w}}_2+\tilde v_2)+(1-\xi)(\tilde{\mathbf{w}}_1+\tilde v_1)}\right) + \Delta\tilde{\mathbf{w}}_2. \end{cases} \tag{49}$$

We fix $\nu \in (-1, 0)$, to solve (49), it is enough to find $(\tilde{v}_1, \tilde{v}_2) \in \left(\mathcal{C}_\nu^{2,\beta}\left(\bar{\Omega}^*(\tilde{\mathbf{q}})\right)\right)^2$ solution of

$$\left\{\begin{array}{l} \tilde{v}_1 = \widetilde{\mathcal{K}}_\nu \circ \tilde{\mathscr{E}}_{r_\varepsilon,\alpha,\tilde{\mathbf{q}}} \circ \tilde{S}_1(\tilde{v}_1, \tilde{v}_2) \\ \tilde{v}_2 = \widetilde{\mathcal{K}}_\nu \circ \tilde{\mathscr{E}}_{r_\varepsilon,\tilde{\mathbf{q}}} \circ \tilde{S}_2(\tilde{v}_1, \tilde{v}_2), \end{array}\right. \tag{50}$$

where

$$\left\{\begin{array}{ll} \tilde{S}_1(\tilde{v}_1,\, \tilde{v}_2) & = \rho^2|z-p_1|^{2\alpha_1}|z-p_2|^{2\alpha_2} f_1(z)\left(e^{\gamma(\tilde{\mathbf{w}}_1+\tilde{v}_1)+(1-\gamma)(\tilde{\mathbf{w}}_2+\tilde{v}_2)}\right) + \Delta\tilde{\mathbf{w}}_1 \\ \tilde{S}_2(\tilde{v}_1,\, \tilde{v}_2) & = \rho^2|z-p_3|^{2\alpha_3}|z-p_2|^{2\alpha_2} f_2(z)\left(e^{\xi(\tilde{\mathbf{w}}_2+\tilde{v}_2)+(1-\xi)(\tilde{\mathbf{w}}_1+\tilde{v}_1)}\right) + \Delta\tilde{\mathbf{w}}_2 \end{array}\right. \tag{51}$$

and for all $\sigma \in (0, r_0/2)$

$$\tilde{\mathscr{E}}_{\sigma,\tilde{q}} : \mathcal{C}_\nu^{0,\beta}(\bar{\Omega}_\sigma(\tilde{q})) \longrightarrow \mathcal{C}_\nu^{0,\beta}(\bar{\Omega}^*(\tilde{q}))$$

the extension operator.

We summarize this in the following proposition.

Proposition

Given $\kappa > 0$, $\delta \in (0,1), \mu \in (1/(2+\bar{\alpha}), \delta)$ and assume that the condition (18) is satisfied, then there exists $\varepsilon_\kappa > 0$ (depending on κ), such that for any $\varepsilon \in (0, \varepsilon_\kappa)$, λ_j satisfying (47), any functions $\tilde{\varphi}_j^i$ satisfying (37) and (47), there exists a unique $(\tilde{v}_1, \tilde{v}_2)$ solution of (50) so that for $\mathbf{v_k}$ $(k=1,2)$ defined by

$$\begin{cases} \mathbf{v_1} := \dfrac{1+\lambda_1+\alpha_1}{\gamma} G(\cdot, p_1) + (1+\lambda_2+\alpha_2) G(\cdot, p_2) + \displaystyle\sum_{i=1}^{2} \chi_{r_0}(\cdot - p_i) H^{ext}(\tilde{\varphi}_1^i; (\cdot - p_i)/r_{\varepsilon,\alpha}) \\ \qquad + \chi_{r_0}(\cdot - \tilde{z}_3) H^{ext}(\tilde{\varphi}_1^3; (\cdot - \tilde{z}_3)/r_\varepsilon) + \tilde{v}_1 \\ \mathbf{v_2} := \dfrac{1+\ \lambda_3}{\xi} G(\cdot, \tilde{z}_3) + (1+\lambda_2+\alpha_2) G(\cdot, p_2) + \displaystyle\sum_{i=1}^{2} \chi_{r_0}(\cdot - p_i) H^{ext}(\tilde{\varphi}_2^i; (\cdot - p_i)/r_{\varepsilon,\alpha}) \\ \qquad + \chi_{r_0}(\cdot - \tilde{z}_3) H^{ext}(\tilde{\varphi}_2^3; (\cdot - \tilde{z}_3)/r_\varepsilon) + \tilde{v}_2, \end{cases}$$

solve (48) in $\bar{\Omega} - \{B_{r_{\varepsilon,\alpha}}(p_1) \cup B_{r_{\varepsilon,\alpha}}(p_2) \cup B_{r_\varepsilon}(\tilde{z}_3)\}$. In addition, we have

$$\|\tilde{v}_k\|_{C_\nu^{2,\beta}(\bar{\Omega}^*(\tilde{\mathbf{q}}))} \leqslant 2\bar{c}_\kappa r_{\varepsilon,\alpha}^2.$$

The nonlinear Cauchy-data matching

It remains to determine the parameters and the functions in such a way that the function which is equal to u_{int} in $B_{r_{\varepsilon,\alpha}}(p_1) \cup B_{r_{\varepsilon,\alpha}}(p_2) \cup B_{r_\varepsilon}(\tilde{z}_3)$ and that is equal to u_{ext} in $\bar{\Omega} - \{B_{r_{\varepsilon,\alpha}}(p_1) \cup B_{r_{\varepsilon,\alpha}}(p_2) \cup B_{r_\varepsilon}(\tilde{z}_3)\ \}$ is a smooth function.
This amounts to find the boundary data and the parameters so that, for each $i = 1, 2$

$$\left\{ \begin{array}{rcl} (u_{int,\,i} - u_{ext,\,i})(p_1 + r_{\varepsilon,\alpha}\,.) & = & 0 \\ (\partial_r u_{int,\,i} - \partial_r u_{ext,\,i})(p_1 + r_{\varepsilon,\alpha}\,.) & = & 0 \end{array} \right. \tag{52}$$

near $\partial B_{r_\varepsilon,\alpha}(p_1)$. Then the systems found by projecting (52) gather in these equalities

$$T^1_{1,\varepsilon,\alpha} = \left(t_1, \lambda_1, \varphi^1_{1,0}, \tilde{\varphi}^1_{1,0}, \varphi^1_{1,\perp}, \tilde{\varphi}^1_{1,\perp}\right) = \mathcal{O}(r_{\varepsilon,\alpha})$$

and

$$T^1_{2,\varepsilon,\alpha} = \left(\varphi^1_{2,0}, \tilde{\varphi}^1_{2,0}, \varphi^1_{2,\perp}, \tilde{\varphi}^1_{2,\perp}\right) = \mathcal{O}(r_{\varepsilon,\alpha}),$$

where

$$t_1 = \frac{1}{\log r_{\varepsilon,\alpha}}\left[2(1+\alpha_1)\log\tau_1 - 2\log(1+\alpha_1) + \log\gamma + \mathcal{E}_1(p_1, \tilde{\mathbf{q}})\right]$$

and

$$\mathcal{E}_1(z,\mathbf{q}) := (1+\alpha_1)H(z,p_1) + \frac{1-\gamma}{\xi}G(z,z_3) + (1+\alpha_2)G(z,p_2) + \log\left(|p_1 - p_2|^{2\alpha_2} f_1(p_1)\right).$$

Near $\partial B_{r_{\varepsilon,\alpha}}(p_2)$, it amounts to find the boundary data and the parameters so that

$$\left\{ \begin{array}{rcl} (u_{int,\,i} - u_{ext,\,i})(p_2 + r_{\varepsilon,\alpha}\cdot) & = & 0 \\ (\partial_r u_{int,\,i} - \partial_r u_{ext,\,i})(p_2 + r_{\varepsilon,\alpha}\cdot) & = & 0. \end{array} \right. \tag{53}$$

Then the systems found by projecting (53) gather in these equalities

$$T^2_{i,\varepsilon,\alpha} = \left(t_2, \lambda_2, \varphi^2_{i,0}, \tilde{\varphi}^2_{i,0}, \varphi^2_{i,\perp}, \tilde{\varphi}^2_{i,\perp}\right) = \mathcal{O}(r_{\varepsilon,\alpha}), \quad \text{for } i = 1, 2,$$

where

$$t_2 = \frac{1}{\log r_{\varepsilon,\alpha}} \left[2(1+\alpha_2)\log \tau_2 - 2\log(1+\alpha_2) + \mathcal{E}_2(p_2, \tilde{\mathbf{q}})\right]$$

and

$$\mathcal{E}_2(z,\mathbf{q}) := (1+\alpha_2)H(z,p_2) - \frac{1-\xi}{\gamma+\xi-1}\log\left(|p_2 - p_1|^{2\alpha_1} f_1(p_2)\right)$$
$$+ \frac{\gamma}{\gamma+\xi-1}\log\left(|p_2 - p_3|^{2\alpha_3} f_2(p_2)\right) + \frac{1}{\xi}G(z, z_3).$$

Near $\partial B_{r_\varepsilon}(\tilde{z}_3)$, it amounts to find the boundary data and the parameters so that

$$\begin{cases} (u_{int,\,i} - u_{ext,\,i})(\tilde{z}_3 + r_\varepsilon \cdot) & = & 0 \\ (\partial_r u_{int,\,i} - \partial_r u_{ext,\,i})(\tilde{z}_3 + r_\varepsilon \cdot) & = & 0. \end{cases} \tag{54}$$

Then the systems found by projecting (54) gather in these equalities

$$T^3_{1,\varepsilon} = \left(\varphi^3_{1,0}, \tilde{\varphi}^3_{1,0}, \varphi^3_{1,1}, \tilde{\varphi}^3_{1,1}, \varphi^3_{1,\perp}, \tilde{\varphi}^3_{1,\perp}\right) = \mathcal{O}(r^2_{\varepsilon,\alpha})$$

and

$$T^3_{2,\varepsilon} = \left(t_3, \lambda_3, \varphi^3_{2,0}, \tilde{\varphi}^3_{2,0}, \varphi^3_{2,1}, \tilde{\varphi}^3_{2,1}, \bar{\nabla}\mathcal{E}_3(\tilde{z}_3, \tilde{\mathbf{q}}), \varphi^3_{2,\perp}, \tilde{\varphi}^3_{2,\perp}\right) = \mathcal{O}(r^2_{\varepsilon,\alpha}),$$

where

$$\mathcal{E}_3(z, \tilde{\mathbf{q}}) := H(z, \tilde{z}_3) + \tfrac{(1+\alpha_1)(1-\xi)}{\gamma} G(z, p_1) + (1+\alpha_2) G(z, p_2)$$
$$+ \log(|\tilde{z}_3 - p_3|^{2\alpha_3} |\tilde{z}_3 - p_2|^{2\alpha_2} f_2(\tilde{z}_3))$$

and

$$t_3 = \frac{1}{\log r_\varepsilon} \left[2 \log \tau_3 + \log \xi + \mathcal{E}_3(\tilde{z}_3, \tilde{\mathbf{q}})\right].$$

Sketch of the proof of Theorem 4

The case : $z_1 = p_1$, $z_2 \neq p_2$ and $z_3 \neq p_3$

- Rotationally symetric solutions
- The nonlinear interior problem
- The nonlinear exterior Problem
- The nonlinear Cauchy-data matching

Rotationally symmetric solutions

We define an ansatz for solution of (38), let $\tau_i \geq 0$, for $i = 1, 2, 3$, then we define

$$\widetilde{u}_1(z) = \begin{cases} \frac{1}{\gamma} u_{\varepsilon,\tau_1,\alpha_1}(z - p_1) - \frac{1-\gamma}{\gamma\xi} G(z, z_3) - \frac{1-\gamma}{\gamma} G(z, z_2) & \\ \quad -\frac{1}{\gamma} \log\left(\gamma |p_1 - p_2|^{2\alpha_2} f_1(p_1)\right) & z \in B(p_1, r_{\varepsilon,\alpha}) \\ u_{\varepsilon,\tau_2}(z - z_2) - \frac{\xi}{\gamma+\xi-1} \log\left(|z_2 - p_1|^{2\alpha_1} |z_2 - p_2|^{2\alpha_2} f_1(z_2)\right) & \\ \quad + \frac{1-\gamma}{\gamma+\xi-1} \log(|z_2 - p_3|^{2\alpha_3} |z_2 - p_2|^{2\alpha_2} f_2(z_2)) & z \in B(z_2, r_\varepsilon) \\ \frac{1+\alpha_1}{\gamma} G(z, p_1) + G(z, z_2) & z \in \Omega \backslash \left(B(p_1, r_{\varepsilon,\alpha}) \cup B(z_2, r_\varepsilon)\right) \end{cases}$$

$$\widetilde{u}_2(z) = \begin{cases} \frac{1}{\xi} u_{\varepsilon,\tau_3}(z - z_3) - \frac{(1+\alpha_1)(1-\xi)}{\gamma\xi} G(z, p_1) - \frac{1-\xi}{\xi} G(z, z_2) & \\ \quad -\frac{1}{\xi} \log(\xi |z_3 - p_3|^{2\alpha_3} |z_3 - p_2|^{2\alpha_2} f_2(z_3)) & z \in B(z_3, r_\varepsilon) \\ u_{\varepsilon,\tau_2}(z - z_2) + \frac{1-\xi}{\gamma+\xi-1} \log\left(|z_2 - p_1|^{2\alpha_1} |z_2 - p_2|^{2\alpha_2} f_1(z_2)\right) & \\ \quad -\frac{\gamma}{\gamma+\xi-1} \log(|z_2 - p_3|^{2\alpha_3} |z_2 - p_2|^{2\alpha_2} f_2(z_2)) & z \in B(z_2, r_\varepsilon) \\ \frac{1}{\xi} G(z, z_3) + G(z, z_2) & z \in \Omega \backslash \left(B(z_2, r_\varepsilon) \cup B(z_3, r_\varepsilon)\right). \end{cases}$$

Then, for $r = |z - p_1|$, we fix $\delta \in \Big(0, \min\left\{ 1,\ 2(1+\alpha_1)(\gamma+\xi-1)/\gamma \right\}\Big)$ and $\mu \in (0,1)$, we have

$$\Big\| \Delta \widetilde{u}_1 + \rho^2 |z-p_1|^{2\alpha_1} |z-p_2|^{2\alpha_2} f_1(z)\ e^{\gamma \widetilde{u}_1 + (1-\gamma)\widetilde{u}_2} \Big\|_{\mathcal{C}^{0,\beta}_{\mu-2}(B_{r_{\varepsilon,\alpha}}(p_1))} \leq C r_{\varepsilon,\alpha}$$

$$\Big\| \Delta \widetilde{u}_2 + \rho^2 |z-p_3|^{2\alpha_3} |z-p_2|^{2\alpha_2} f_2(z)\ e^{\xi \widetilde{u}_2 + (1-\xi)\widetilde{u}_1} \Big\|_{\mathcal{C}^{0,\beta}_{\delta-2}(B_{r_{\varepsilon,\alpha}}(p_1))} \leq C r_{\varepsilon,\alpha}.$$

Then, for $r = |z - z_2|$, we fix $\mu \in (1,2)$ and using the condition (25), we have

$$\Big\| \Delta \widetilde{u}_1 + \rho^2 |z-p_1|^{2\alpha_1} |z-p_2|^{2\alpha_2} f_1(z)\ e^{\gamma \widetilde{u}_1 + (1-\gamma)\widetilde{u}_2} \Big\|_{\mathcal{C}^{0,\beta}_{\mu-2}(B_{r_\varepsilon}(z_2))} \leq C r_\varepsilon^2$$

$$\Big\| \Delta \widetilde{u}_2 + \rho^2 |z-p_3|^{2\alpha_3} |z-p_2|^{2\alpha_2} f_2(z)\ e^{\xi \widetilde{u}_2 + (1-\xi)\widetilde{u}_1} \Big\|_{\mathcal{C}^{0,\beta}_{\mu-2}(B_{r_\varepsilon}(z_2))} \leq C r_\varepsilon^2.$$

Then, for $r = |z - z_3|$, we fix $\delta \in (0,\ \frac{\gamma+\xi-1}{\xi})$, $\mu \in (1,2)$ and using the condition (26), we have

$$\Big\| \Delta \widetilde{u}_1 + \rho^2 |z-p_1|^{2\alpha_1} |z-p_2|^{2\alpha_2} f_1(z)\ e^{\gamma \widetilde{u}_1 + (1-\gamma)\widetilde{u}_2} \Big\|_{\mathcal{C}^{0,\beta}_{\delta-2}(B_{r_\varepsilon}(z_3))} \leq C r_\varepsilon^2$$

$$\Big\| \Delta \widetilde{u}_2 + \rho^2 |z-p_3|^{2\alpha_3} |z-p_2|^{2\alpha_2} f_2(z)\ e^{\xi \widetilde{u}_2 + (1-\xi)\widetilde{u}_1} \Big\|_{\mathcal{C}^{0,\beta}_{\mu-2}(B_{r_\varepsilon}(z_3))} \leq C r_\varepsilon^2.$$

The nonlinear interior problem

• In $\mathbf{B}(\mathbf{p_1}, \mathbf{r}_{\varepsilon,\alpha})$: we obtain the following proposition.

Proposition

Given $\kappa > 0$, $\alpha_1 \notin \mathbb{N}$, $\delta \in (0, \delta_1)$ and $\mu \in (0, \delta)$ there exist $\varepsilon_\kappa > 0$ and $c_\kappa > 0$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, there exists a unique (h_1^1, h_2^1) solution of (40) such that $\|(h_1^1, h_2^1)\|_{\mathcal{C}_\mu^{2,\beta}(B_{r_{\varepsilon,\alpha}}(p_1)) \times \mathcal{C}_\delta^{2,\beta}(B_{r_{\varepsilon,\alpha}}(p_1))} \leqslant c_\kappa r_{\varepsilon,\alpha}$. Hence

$$\begin{cases} v_1(z) = & \tilde{u}_1(z) + H^{int}(\varphi_1^1;\ (z - p_1)/r_{\varepsilon,\alpha}) + \ h_1^1(z) \\ v_2(z) = & \tilde{u}_2(z) + H^{int}(\varphi_2^1;\ (z - p_1)/r_{\varepsilon,\alpha}) + h_2^1(z). \end{cases}$$

solves (38) in $B_{r_{\varepsilon,\alpha}}(p_1)$.

• **In $\mathbf{B(z_2, r_\varepsilon)}$**: Using the following transformation

$$\begin{cases} v_1(z) &= u_1(\frac{\varepsilon}{\tau_2}z) + \frac{4}{\gamma}\ \log\varepsilon - \frac{2}{\gamma}\ \log(\frac{\tau_2(1+\varepsilon^2)}{2}) \\ v_2(z) &= u_2(\frac{\varepsilon}{\tau_2}z) + \frac{4}{\xi}\ \log\varepsilon - \frac{2}{\xi}\ \log(\frac{\tau_2(1+\varepsilon^2)}{2}). \end{cases} \tag{55}$$

Then the system (38) can be written as

$$\begin{cases} -\Delta v_1 &= 2|\frac{\varepsilon}{\tau_2}z - p_1|^{2\alpha_1}|\frac{\varepsilon}{\tau_2}z - p_2|^{2\alpha_2} f_1(\frac{\varepsilon}{\tau_2}z)e^{\gamma v_1 + (1-\gamma)v_2} \quad \text{in } B_{R_\varepsilon^2}(z_2) \\ -\Delta v_2 &= 2|\frac{\varepsilon}{\tau_2}z - p_3|^{2\alpha_3}|\frac{\varepsilon}{\tau_2}z - p_2|^{2\alpha_2} f_2(\frac{\varepsilon}{\tau_2}z)e^{\xi v_2 + (1-\xi)v_1} \quad \text{in } B_{R_\varepsilon^2}(z_2), \end{cases} \tag{56}$$

where $R_\varepsilon^2 := \dfrac{\tau_2 r_\varepsilon}{\varepsilon}$.

In $B(z_2, R_\varepsilon^2)$: We look for a solution of (69) of the form

$$\begin{cases} v_1(z) = u_{\varepsilon,\tau_2}(z - z_2) - \frac{\xi}{\gamma+\xi-1} \log\left(|\frac{\varepsilon}{\tau_2} z_2 - p_1|^{2\alpha_1} |\frac{\varepsilon}{\tau_2} z_2 - p_2|^{2\alpha_2} f_1(\frac{\varepsilon}{\tau_2} z_2)\right) \\ \qquad + \frac{1-\gamma}{\gamma+\xi-1} \log\left(|\frac{\varepsilon}{\tau_2} z_2 - p_3|^{2\alpha_3} |\frac{\varepsilon}{\tau_2} z_2 - p_2|^{2\alpha_2} f_2(\frac{\varepsilon}{\tau_2} z_2)\right) + H^{int}(\varphi_1^2;\ (z - z_2)/R_\varepsilon^2) + \ h_1^2(z) \\ v_2(z) = u_{\varepsilon,\tau_2}(z - z_2) + \frac{1-\xi}{\gamma+\xi-1} \log\left(|\frac{\varepsilon}{\tau_2} z_2 - p_1|^{2\alpha_1} |\frac{\varepsilon}{\tau_2} z_2 - p_2|^{2\alpha_2} f_1(\frac{\varepsilon}{\tau_2} z_2)\right) \\ \qquad - \frac{\gamma}{\gamma+\xi-1} \log\left(|\frac{\varepsilon}{\tau_2} z_2 - p_3|^{2\alpha_3} |\frac{\varepsilon}{\tau_2} z_2 - p_2|^{2\alpha_2} f_2(\frac{\varepsilon}{\tau_2} z_2)\right) + H^{int}(\varphi_2^2;\ (z - z_2)/R_\varepsilon^2) + h_2^2(z). \end{cases} \tag{57}$$

This amounts to solve the equations

$$\begin{cases} \mathbb{L} h_1^2 = \frac{8}{(1+r^2)^2} \left[\frac{|\frac{\varepsilon}{\tau_2} z - p_1|^{2\alpha_1} |\frac{\varepsilon}{\tau_2} z - p_2|^{2\alpha_2} f_1(\frac{\varepsilon}{\tau_2} z)}{|\frac{\varepsilon}{\tau_2} z_2 - p_1|^{2\alpha_1} |\frac{\varepsilon}{\tau_2} z_2 - p_2|^{2\alpha_2} f_1(\frac{\varepsilon}{\tau_2} z_2)} e^{\gamma\left(h_1^2 + H_1^{int,2}\right) + (1-\gamma)\left(h_2^2 + H_2^{int,2}\right)} - h_1^2 - 1 \right] \\ \mathbb{L} h_2^2 = \frac{8}{(1+r^2)^2} \left[\frac{|\frac{\varepsilon}{\tau_2} z - p_3|^{2\alpha_3} |\frac{\varepsilon}{\tau_2} z - p_2|^{2\alpha_2} f_2(\frac{\varepsilon}{\tau_2} z)}{|\frac{\varepsilon}{\tau_2} z_2 - p_3|^{2\alpha_3} |\frac{\varepsilon}{\tau_2} z_2 - p_2|^{2\alpha_2} f_2(\frac{\varepsilon}{\tau_2} z_2)} e^{\xi\left(h_2^2 + H_2^{int,2}\right) + (1-\xi)\left(h_1^2 + H_1^{int,2}\right)} - h_2^2 - 1 \right]. \end{cases} \tag{58}$$

We denote by

$$\mathbb{L}h_1^2 = \mathcal{R}_3(h_1^2, h_2^2) \quad \text{and} \quad \mathbb{L}h_2^2 = \mathcal{R}_4(h_1^2, h_2^2).$$

To find a solution of (58), it is enough to find a fixed point (h_1^2, h_2^2) in a small ball of $\mathcal{C}_\mu^{2,\beta}(\mathbb{R}^2) \times \mathcal{C}_\mu^{2,\beta}(\mathbb{R}^2)$ solutions of

$$\begin{cases} h_1^2 &= \mathcal{G}_\mu \circ \mathscr{E}_\mu \circ \mathcal{R}_3(h_1^2, h_2^2) = \mathcal{N}_2(h_1^2, h_2^2) \\ h_2^2 &= \mathcal{G}_\mu \circ \mathscr{E}_\mu \circ \mathcal{R}_4(h_1^2, h_2^2) = \mathcal{M}_2(h_1^2, h_2^2). \end{cases} \tag{59}$$

We summarize this in the following proposition.

Proposition

Given $\kappa > 0$, $\alpha_2 \notin \mathbb{N}$, γ_0, ξ_0 and μ in $(0,1)$. Assume that the condition (25) is satisfied, then there exist $\varepsilon_\kappa > 0$ and $c_\kappa > 0$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\gamma \in (\gamma_0, 1)$ and $\xi \in (\xi_0, 1)$, there exists a unique (h_1^2, h_2^2) solution of (58) such that

$$\|(h_1^2, h_2^2)\|_{\mathcal{C}_\mu^{2,\beta}(\mathbb{R}^2)\times\mathcal{C}_\mu^{2,\beta}(\mathbb{R}^2)} \leqslant c_\kappa r_\varepsilon^2.$$

Hence (v_1, v_2) defined by (44) solves (56) in $B_{R_\varepsilon^2}(z_2)$.

We denote by

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To find a solution of (58), it is enough to find a fixed point (h_1^2, h_2^2) in a small ball of $\mathcal{C}_\mu^{2,\beta}(\mathbb{R}^2) \times \mathcal{C}_\mu^{2,\beta}(\mathbb{R}^2)$ solutions of

$$\begin{cases} h_1^2 & = & \mathcal{G}_\mu \circ \mathscr{E}_\mu \circ \mathcal{R}_3(h_1^2, h_2^2) = \mathcal{N}_2(h_1^2, h_2^2) \\ h_2^2 & = & \mathcal{G}_\mu \circ \mathscr{E}_\mu \circ \mathcal{R}_4(h_1^2, h_2^2) = \mathcal{M}_2(h_1^2, h_2^2). \end{cases} \tag{59}$$

We summarize this in the following proposition.

Proposition

Given $\kappa > 0$, $\alpha_2 \notin \mathbb{N}$, γ_0, ξ_0 and μ in $(0,1)$. Assume that the condition (25) is satisfied, then there exist $\varepsilon_\kappa > 0$ and $c_\kappa > 0$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\gamma \in (\gamma_0, 1)$ and $\xi \in (\xi_0, 1)$, there exists a unique (h_1^2, h_2^2) solution of (58) such that

$$\|(h_1^2, h_2^2)\|_{\mathcal{C}_\mu^{2,\beta}(\mathbb{R}^2) \times \mathcal{C}_\mu^{2,\beta}(\mathbb{R}^2)} \leqslant c_\kappa r_\varepsilon^2.$$

Hence (v_1, v_2) defined by (44) solves (56) in $B_{R_\varepsilon^2}(z_2)$.

- In $\mathbf{B}(\mathbf{z_3}, \mathbf{R}^3_\varepsilon)$: We obtain the following proposition.

Proposition

Given $\kappa > 0$, $\mu \in (1,2)$, $\delta \in \left(0, \frac{\gamma+\xi-1}{\xi}\right)$ *and assume that the condition* (26) *is satisfied, then there exist* $\varepsilon_\kappa > 0$, $c_\kappa > 0$ *and* $\xi_0 \in (0,1)$ *such that for all* $\varepsilon \in (0, \varepsilon_\kappa)$, $\xi \in (\xi_0, 1)$, *there exists a unique* (h_1^3, h_2^3) *solution of* (46) *such that* $\|(h_1^3, h_2^3)\|_{C^{2,\beta}_\delta(\mathbb{R}^2)\times C^{2,\beta}_\mu(\mathbb{R}^2)} \leq 2c_\kappa r_\varepsilon^2$. *Hence*

$$\begin{cases} v_1(z) = \frac{1+\alpha_1}{\gamma} G(\frac{\varepsilon}{\tau_3} z, p_1) + G(\frac{\varepsilon}{\tau_3} z, z_2) + H^{int}_{\varphi_1^3}(\frac{z-z_3}{R^3_\varepsilon}) + h_1^3(z) \\ v_2(z) = \frac{1}{\xi} u_{\varepsilon,\tau_3}(z-z_3) - \frac{(1-\xi)(1+\alpha_1)}{\gamma\xi} G(\frac{\varepsilon}{\tau_3} z, p_1) - \frac{1-\xi}{\xi} G(\frac{\varepsilon}{\tau_3} z, z_2) \\ \qquad - \frac{1}{\xi} \log\left(\xi |\frac{\varepsilon}{\tau_3} z_3 - p_3|^{2\alpha_3} |\frac{\varepsilon}{\tau_3} z_3 - p_2|^{2\alpha_2} f_2(\frac{\varepsilon}{\tau_3} z_3)\right) + H^{int}_{\varphi_2^3}(\frac{z-z_3}{R^3_\varepsilon}) + h_2^3(z), \end{cases}$$

solves (44) *in* $B_{R^3_\varepsilon}(z_3)$.

The nonlinear exterior problem

Given $\tilde{\mathbf{P}} := (p_1, \tilde{z}_2, \tilde{z}_3) \in \Omega^3$ close to $\mathbf{P} := (p_1, z_2, z_3)$, $\lambda := (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$ close to 0 and $(\tilde{\varphi}_1^i, \tilde{\varphi}_2^i) \in (\mathcal{C}^{2,\beta}(S^1))^2$, $i = 1, 2, 3$ satisfying (37). Define

$$\begin{cases} \tilde{\mathbf{w}}_1 & := & \dfrac{1+\lambda_1+\alpha_1}{\gamma} G(\cdot, p_1) + (1+\lambda_2) G(\cdot, \tilde{z}_2) + \chi_{r_0}(\cdot - p_1) H_1^{ext}(\tilde{\varphi}_1^1; (\cdot - p_1)/r_{\varepsilon,\alpha}) \\ & & + \chi_{r_0}(\cdot - \tilde{z}_2) H_1^{ext}(\tilde{\varphi}_1^2; (\cdot - \tilde{z}_2)/r_\varepsilon) + \chi_{r_0}(\cdot - \tilde{z}_3) H_1^{ext}(\tilde{\varphi}_1^3; (\cdot - \tilde{z}_3)/r_\varepsilon) \\ \tilde{\mathbf{w}}_2 & := & \dfrac{1+\lambda_3}{\xi} G(\cdot, \tilde{z}_3) + (1+\lambda_2) G(\cdot, \tilde{z}_2) + \chi_{r_0}(\cdot - p_1) H_2^{ext}(\tilde{\varphi}_2^1; (\cdot - p_1)/r_{\varepsilon,\alpha}) \\ & & + \chi_{r_0}(\cdot - \tilde{z}_2) H_2^{ext}(\tilde{\varphi}_2^2; (\cdot - \tilde{z}_2)/r_\varepsilon) + \chi_{r_0}(\cdot - \tilde{z}_3) H_2^{ext}(\tilde{\varphi}_2^3; (\cdot - \tilde{z}_3)/r_\varepsilon). \end{cases}$$

Here χ_{r_0} is a cut-off function identically equal to 1 in $B_{r_0/2}$ and identically equal to 0 outside B_{r_0}.

We would like to find a solution of the system

$$\begin{cases} -\Delta u_1 &= \rho^2|z-p_1|^{2\alpha_1}|z-p_2|^{2\alpha_2}f_1(z)e^{\gamma u_1+(1-\gamma)u_2} \\ -\Delta u_2 &= \rho^2|z-p_3|^{2\alpha_3}|z-p_2|^{2\alpha_2}f_2(z)e^{\xi u_2+(1-\xi)u_1} \end{cases} \tag{60}$$

which is defined in $\bar{\Omega}-\{B_{r_{\varepsilon,\alpha}}(p_1)\cup B_{r_\varepsilon}(\tilde{z}_2)\cup B_{r_\varepsilon}(\tilde{z}_3)\}$, with $u_1=\tilde{\mathbf{w}}_1+\tilde{v}_1$ a perturbation of $\tilde{\mathbf{w}}_1$ and $u_2=\tilde{\mathbf{w}}_2+\tilde{v}_2$ a perturbation of $\tilde{\mathbf{w}}_2$.

This amounts to solve in $\bar{\Omega}-\{B_{r_{\varepsilon,\alpha}}(p_1)\cup B_{r_\varepsilon}(\tilde{z}_2)\cup B_{r_\varepsilon}(\tilde{z}_3)\}$

$$\begin{cases} -\Delta\tilde{v}_1 &= \rho^2|z-p_1|^{2\alpha_1}|z-p_2|^{2\alpha_2}f_1(z)\left(e^{\gamma(\tilde{\mathbf{w}}_1+\tilde{v}_1)+(1-\gamma)(\tilde{\mathbf{w}}_2+\tilde{v}_2)}\right)+\Delta\tilde{\mathbf{w}}_1 \\ -\Delta\tilde{v}_2 &= \rho^2|z-p_3|^{2\alpha_3}|z-p_2|^{2\alpha_2}f_2(z)\left(e^{\xi(\tilde{\mathbf{w}}_2+\tilde{v}_2)+(1-\xi)(\tilde{\mathbf{w}}_1+\tilde{v}_1)}\right)+\Delta\tilde{\mathbf{w}}_2. \end{cases} \tag{61}$$

We fix $\nu \in (-1, 0)$, to solve (61), it is enough to find $(\tilde{v}_1, \tilde{v}_2) \in \left(\mathcal{C}^{2,\beta}_\nu\left(\bar{\Omega}^*(\tilde{\mathbf{p}})\right)\right)^2$ solution of

$$\tilde{v}_1 = \widetilde{\mathcal{K}}_\nu \circ \tilde{\mathscr{E}}_{r_\varepsilon, \tilde{\mathbf{p}}} \circ \tilde{S}_1(\tilde{v}_1, \tilde{v}_2) \quad \text{and} \quad \tilde{v}_2 = \widetilde{\mathcal{K}}_\nu \circ \tilde{\mathscr{E}}_{r_\varepsilon, \tilde{\mathbf{p}}} \circ \tilde{S}_2(\tilde{v}_1, \tilde{v}_2), \tag{62}$$

where

$$\begin{cases} \tilde{S}_1(\tilde{v}_1,\, \tilde{v}_2) & = \rho^2 |z - p_1|^{2\alpha_1} |z - p_2|^{2\alpha_2} f_1(z) \left(e^{\gamma(\tilde{\mathbf{w}}_1 + \tilde{v}_1) + (1-\gamma)(\tilde{\mathbf{w}}_2 + \tilde{v}_2)}\right) + \Delta \tilde{\mathbf{w}}_1 \\ \tilde{S}_2(\tilde{v}_1,\, \tilde{v}_2) & = \rho^2 |z - p_3|^{2\alpha_3} |z - p_2|^{2\alpha_2} f_2(z) \left(e^{\xi(\tilde{\mathbf{w}}_2 + \tilde{v}_2) + (1-\xi)(\tilde{\mathbf{w}}_1 + \tilde{v}_1)}\right) + \Delta \tilde{\mathbf{w}}_2 \end{cases} \tag{63}$$

and for all $\sigma \in (0, r_0/2)$

$$\tilde{\mathscr{E}}_{\sigma, \tilde{p}} : \mathcal{C}^{0,\beta}_\nu(\bar{\Omega}_\sigma(\tilde{p})) \longrightarrow \mathcal{C}^{0,\beta}_\nu(\bar{\Omega}^*(\tilde{p}))$$

the extension operator.

Given $\kappa > 0$, (whose value will be fixed later on), we further assume that, for all $k = 1, 2$ and $i = 1, 2, 3$,

$$\|\tilde{\varphi}^i_k\|_{\mathcal{C}^{2,\beta}} \leqslant \kappa r^2_{\varepsilon,\alpha}, \quad |\lambda_i| \leqslant r^2_{\varepsilon,\alpha}, \quad |z_2 - \tilde{z}_2| \leqslant \kappa r_\varepsilon \quad \text{and} \quad |z_3 - \tilde{z}_3| \leqslant \kappa r_\varepsilon. \tag{64}$$

We fix $\nu \in (-1,0)$, to solve (61), it is enough to find $(\tilde v_1, \tilde v_2) \in \left(\mathcal{C}^{2,\beta}_\nu\left(\bar\Omega^*(\tilde{\mathbf{p}})\right)\right)^2$ solution of

$$\tilde v_1 = \widetilde{\mathcal{K}}_\nu \circ \tilde{\mathscr{E}}_{r_\varepsilon,\tilde{\mathbf{p}}} \circ \tilde S_1(\tilde v_1, \tilde v_2) \quad \text{and} \quad \tilde v_2 = \widetilde{\mathcal{K}}_\nu \circ \tilde{\mathscr{E}}_{r_\varepsilon,\tilde{\mathbf{p}}} \circ \tilde S_2(\tilde v_1, \tilde v_2), \tag{62}$$

where

$$\begin{cases} \tilde S_1(\tilde v_1,\, \tilde v_2) & = \rho^2 |z-p_1|^{2\alpha_1}|z-p_2|^{2\alpha_2} f_1(z)\,\left(e^{\gamma(\tilde{\mathbf{w}}_1+\tilde v_1)+(1-\gamma)(\tilde{\mathbf{w}}_2+\tilde v_2)}\right) + \Delta\tilde{\mathbf{w}}_1 \\ \tilde S_2(\tilde v_1,\, \tilde v_2) & = \rho^2 |z-p_3|^{2\alpha_3}|z-p_2|^{2\alpha_2} f_2(z)\,\left(e^{\xi(\tilde{\mathbf{w}}_2+\tilde v_2)+(1-\xi)(\tilde{\mathbf{w}}_1+\tilde v_1)}\right) + \Delta\tilde{\mathbf{w}}_2 \end{cases} \tag{63}$$

and for all $\sigma \in (0, r_0/2)$

$$\tilde{\mathscr{E}}_{\sigma,\tilde p} : \mathcal{C}^{0,\beta}_\nu(\bar\Omega_\sigma(\tilde p)) \longrightarrow \mathcal{C}^{0,\beta}_\nu(\bar\Omega^*(\tilde p))$$

the extension operator.
Given $\kappa > 0$, (whose value will be fixed later on), we further assume that, for all $k = 1, 2$ and $i = 1, 2, 3$,

$$\|\tilde\varphi^i_k\|_{\mathcal{C}^{2,\beta}} \leqslant \kappa r^2_{\varepsilon,\alpha}, \quad |\lambda_i| \leqslant r^2_{\varepsilon,\alpha}, \quad |z_2 - \tilde z_2| \leqslant \kappa r_\varepsilon \quad \text{and} \quad |z_3 - \tilde z_3| \leqslant \kappa r_\varepsilon. \tag{64}$$

We summarize this in the following proposition.

Proposition

Given $\kappa > 0$, $\delta \in (0,1)$, $\mu \in (1/(2+\bar{\alpha}), \delta)$ and assume that the condition (24) is satisfied, there exists $\varepsilon_\kappa > 0$ (depending on κ) such that for any $\varepsilon \in (0, \varepsilon_\kappa)$, λ_i satisfying (64), any functions $\tilde{\varphi}_j^i$ satisfying (37) and (64), there exists a unique $(\tilde{v}_1, \tilde{v}_2)$ solution of (62) so that for $\mathbf{v_k}$ $(k = 1, 2)$ defined by

$$\begin{cases} \mathbf{v_1} & := \dfrac{1+\lambda_1+\alpha_1}{\gamma} G(\cdot, p_1) + (1+\lambda_2) G(\cdot, \tilde{z}_2) + \chi_{r_0}(\cdot - p_1) H_1^{ext}(\tilde{\varphi}_1^1; (\cdot - p_1)/r_{\varepsilon,\alpha}) \\ & +\chi_{r_0}(\cdot - \tilde{z}_2) H_1^{ext}(\tilde{\varphi}_1^2; (\cdot - \tilde{z}_2)/r_\varepsilon) + \chi_{r_0}(\cdot - \tilde{z}_3) H_1^{ext}(\tilde{\varphi}_1^3; (\cdot - \tilde{z}_3)/r_\varepsilon) + \tilde{v}_1 \\ \mathbf{v_2} & := \dfrac{1+\lambda_3}{\xi} G(\cdot, \tilde{z}_3) + (1+\lambda_2) G(\cdot, \tilde{z}_2) + \chi_{r_0}(\cdot - p_1) H_2^{ext}(\tilde{\varphi}_2^1; (\cdot - p_1)/r_{\varepsilon,\alpha}) \\ & +\chi_{r_0}(\cdot - \tilde{z}_2) H_2^{ext}(\tilde{\varphi}_2^2; (\cdot - \tilde{z}_2)/r_\varepsilon) + \chi_{r_0}(\cdot - \tilde{z}_3) H_2^{ext}(\tilde{\varphi}_2^3; (\cdot - \tilde{z}_3)/r_\varepsilon) + \tilde{v}_2, \end{cases}$$

solve (60) in $\bar{\Omega} - \left\{ B_{r_{\varepsilon,\alpha}}(p_1) \cup B_{r_\varepsilon}(\tilde{z}_2) \cup B_{r_\varepsilon}(\tilde{z}_3) \right\}$. In addition, we have
$\|\tilde{v}_k\|_{C_\nu^{2,\beta}(\bar{\Omega}^*(\tilde{\mathbf{p}}))} \leqslant 2\bar{c}_\kappa r_{\varepsilon,\alpha}^2$.

The nonlinear Cauchy-data matching

It remains to determine the parameters and the functions in such a way that the function which is equal to u_{int} in $B_{r_{\varepsilon,\alpha}}(p_1) \cup B_{r_\varepsilon}(\tilde{z}_2) \cup B_{r_\varepsilon}(\tilde{z}_3)$ and that is equal to u_{ext} in $\bar{\Omega} - \{B_{r_{\varepsilon,\alpha}}(p_1) \cup B_{r_\varepsilon}(\tilde{z}_2) \cup B_{r_\varepsilon}(\tilde{z}_3)\}$ is a smooth function.
This amounts to find the boundary data and the parameters so that, for each $i = 1, 2$

$$\begin{cases} (u_{int,\,i} - u_{ext,\,i})(p_1 + r_{\varepsilon,\alpha}\,\cdot) & = & 0 \\ (\partial_r u_{int,\,i} - \partial_r u_{ext,\,i})(p_1 + r_{\varepsilon,\alpha}\,\cdot) & = & 0 \end{cases} \tag{65}$$

near $\partial B_{r_{\varepsilon,\alpha}}(p_1)$. Then the systems found by projecting (65) gather in these equalities

$$T^1_{1,\varepsilon,\alpha} = \left(t_1, \lambda_1, \varphi^1_{1,0}, \tilde{\varphi}^1_{1,0}, \varphi^1_{1,\perp}, \tilde{\varphi}^1_{1,\perp}\right) = \mathcal{O}(r_{\varepsilon,\alpha}),$$

where

$$t_1 = \frac{1}{\log r_{\varepsilon,\alpha}}\left[2(1+\alpha_1)\log\tau_1 - 2\log(1+\alpha_1) + \log\gamma + \mathcal{E}_1(p_1, \tilde{\mathbf{p}})\right]$$

and

$$T^1_{2,\varepsilon,\alpha} = \left(\varphi^1_{2,0}, \tilde{\varphi}^1_{2,0}, \varphi^1_{2,\perp}, \tilde{\varphi}^1_{2,\perp}\right) = \mathcal{O}(r_{\varepsilon,\alpha}).$$

Near $\partial B_{r_\varepsilon}(\tilde z_2)$, it amounts to find the boundary data and the parameters so that

$$\begin{aligned} &\Big((1-\xi)(u_{int,1}-u_{ext,1})+(1-\gamma)(u_{int,2}-u_{ext,2})\Big)(\tilde z_2+r_\varepsilon\cdot)=0\\ \partial_r&\Big((1-\xi)(u_{int,1}-u_{ext,1})+(1-\gamma)(u_{int,2}-u_{ext,2})\Big)(\tilde z_2+r_\varepsilon\cdot)=0. \end{aligned} \tag{66}$$

Then the systems found by projecting (72) gather in this equality

$$T^2_{c,\varepsilon}=\Big(t_2,\lambda_2,\varphi^2_0,\tilde\varphi^2_0,\varphi^2_1,\tilde\varphi^2_1,\bar\nabla\mathcal{E}_2(\tilde z_2,\tilde{\mathbf{p}}),\varphi^2_\perp,\tilde\varphi^2_\perp\Big)=\mathcal{O}(r^2_{\varepsilon,\alpha}),$$

where

$$\begin{aligned}\mathcal{E}_2(z,\tilde{\mathbf{p}}):=&(2-\gamma-\xi)H(z,\tilde z_2)+\tfrac{(1+\alpha_1)(1-\xi)}{\gamma}G(z,p_1)+\tfrac{1-\gamma}{\xi}G(z,\tilde z_3)\\ &+(1-\xi)\log\left(|\tilde z_2-p_1|^{2\alpha_1}|\tilde z_2-p_2|^{2\alpha_2}f_1(\tilde z_2)\right)+(1-\gamma)\log\left(|\tilde z_2-p_3|^{2\alpha_3}|\tilde z_2-p_2|^{2\alpha_2}f_2(\tilde z_2)\right)\end{aligned}$$

and

$$t_2=\frac{1}{\log r_\varepsilon}\Big[2\log\tau_2+\frac{\mathcal{E}_2(\tilde z_2,\tilde{\mathbf{p}})}{2-\gamma-\xi}\Big].$$

Near $\partial B_{r_\varepsilon}(\tilde{z}_3)$, it amounts to find the boundary data and the parameters so that

$$\left\{ \begin{array}{rcl} (u_{int,\,i} - u_{ext,\,i})(\tilde{z}_3 + r_\varepsilon\,.) & = & 0 \\ (\partial_r u_{int,\,i} - \partial_r u_{ext,\,i})(\tilde{z}_3 + r_\varepsilon\,.) & = & 0. \end{array} \right. \tag{67}$$

Then the systems found by projecting (67) gather in these equalities

$$T^3_{1,\varepsilon} = \left(\varphi^3_{1,0}, \tilde{\varphi}^3_{1,0}, \varphi^3_{1,1}, \tilde{\varphi}^3_{1,1}, \varphi^3_{1,\perp}, \tilde{\varphi}^3_{1,\perp}\right) = \mathcal{O}(r^2_{\varepsilon,\alpha})$$

and

$$T^3_{2,\varepsilon} = \left(t_3, \lambda_3, \varphi^3_{2,0}, \tilde{\varphi}^3_{2,0}, \varphi^3_{2,1}, \tilde{\varphi}^3_{2,1}, \bar{\nabla}\mathcal{E}_3(\tilde{z}_3, \tilde{\mathbf{p}}), \varphi^3_{2,\perp}, \tilde{\varphi}^3_{2,\perp}\right) = \mathcal{O}(r^2_{\varepsilon,\alpha}),$$

where

$$\mathcal{E}_3(z, \tilde{\mathbf{p}}) := H(z, \tilde{z}_3) + \frac{(1+\alpha_1)(1-\xi)}{\gamma} G(z, p_1) + G(z, \tilde{z}_2) + \log\left(|\tilde{z}_3 - p_3|^{2\alpha_3} |\tilde{z}_3 - p_2|^{2\alpha_2} f_2(\tilde{z}_3)\right)$$

and

$$t_3 = \frac{1}{\log r_\varepsilon} \left[2 \log \tau_3 + \log \xi + \mathcal{E}_3(\tilde{z}_3, \tilde{\mathbf{p}})\right].$$

Sketch of the proof of Theorem 5

We define another ansatz for solution of (38), let $\tau_i \geq 0$, for $i = 1, 2, 3$, then we define

$$
\widetilde{u}_1(z) = \begin{cases}
\frac{1}{\gamma} u_{\varepsilon,\tau_1,\alpha_1}(z - p_1) - \frac{1-\gamma}{\gamma\xi} G(z, z_3) - \frac{1-\gamma}{\gamma} G(z, z_2) \\
\qquad - \frac{1}{\gamma} \log(\gamma |p_1 - p_2|^{2\alpha_2} f_1(p_1)) & z \in B(p_1, r_{\varepsilon,\alpha}) \\
u_{\varepsilon,\tau_2}(z - z_2) + \frac{(1+\alpha_1)(1-\gamma)}{\gamma(2-\gamma-\xi)} G(z, p_1) - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G(z, z_3) \\
\qquad - \frac{\xi}{\gamma+\xi-1} \log(|z_2 - p_1|^{2\alpha_1} |z_2 - p_2|^{2\alpha_2} f_1(z_2)) \\
\qquad + \frac{1-\gamma}{\gamma+\xi-1} \log(|z_2 - p_3|^{2\alpha_3} |z_2 - p_2|^{2\alpha_2} f_2(z_2)) & z \in B(z_2, r_\varepsilon) \\
\frac{1+\alpha_1}{\gamma} G(z, p_1) + G(z, z_2) & z \in \Omega \setminus \big(B(p_1, r_{\varepsilon,\alpha}) \cup B(z_2, r_\varepsilon)\big)
\end{cases}
$$

$$
\widetilde{u}_2(z) = \begin{cases}
\frac{1}{\xi} u_{\varepsilon,\tau_3}(z - z_3) - \frac{(1+\alpha_1)(1-\xi)}{\gamma\xi} G(z, p_1) - \frac{1-\xi}{\xi} G(z, z_2) \\
\qquad - \frac{1}{\xi} \log(\xi |z_3 - p_3|^{2\alpha_3} |z_3 - p_2|^{2\alpha_2} f_2(z_3)) & z \in B(z_3, r_\varepsilon) \\
u_{\varepsilon,\tau_2}(z - z_2) - \frac{(1+\alpha_1)(1-\xi)}{\gamma(2-\gamma-\xi)} G(z, p_1) + \frac{1-\xi}{\xi(2-\gamma-\xi)} G(z, z_3) \\
\qquad + \frac{1-\xi}{\gamma+\xi-1} \log(|z_2 - p_1|^{2\alpha_1} |z_2 - p_2|^{2\alpha_2} f_1(z_2)) \\
\qquad - \frac{\gamma}{\gamma+\xi-1} \log(|z_2 - p_3|^{2\alpha_3} |z_2 - p_2|^{2\alpha_2} f_2(z_2)) & z \in B(z_2, r_\varepsilon) \\
\frac{1}{\xi} G(z, z_3) + G(z, z_2) & z \in \Omega \setminus \big(B(z_2, r_\varepsilon) \cup B(z_3, r_\varepsilon)\big).
\end{cases}
$$

Then, for $r = |z - p_1|$, we fix $\delta \in \left(0, \min\left\{1,\, 2(1+\alpha_1)(\gamma+\xi-1)/\gamma\right\}\right)$ and $\mu \in (0,1)$, we have

$$\left\|\Delta \widetilde{u}_1 + \rho^2 |z-p_1|^{2\alpha_1} |z-p_2|^{2\alpha_2} f_1(z)\; e^{\gamma \widetilde{u}_1 + (1-\gamma)\widetilde{u}_2}\right\|_{\mathcal{C}^{0,\beta}_{\mu-2}(B_{r_{\varepsilon,\alpha}}(p_1))} \le C r_{\varepsilon,\alpha}$$

$$\left\|\Delta \widetilde{u}_2 + \rho^2 |z-p_3|^{2\alpha_3} |z-p_2|^{2\alpha_2} f_2(z)\; e^{\xi \widetilde{u}_2 + (1-\xi)\widetilde{u}_1}\right\|_{\mathcal{C}^{0,\beta}_{\delta-2}(B_{r_{\varepsilon,\alpha}}(p_1))} \le C r_{\varepsilon,\alpha}.$$

Then, for $r = |z - z_3|$, we fix $\delta \in (0,\, \frac{\gamma+\xi-1}{\xi})$, $\mu \in (1,2)$ and using the condition (26), we have

$$\left\|\Delta \widetilde{u}_1 + \rho^2 |z-p_1|^{2\alpha_1} |z-p_2|^{2\alpha_2} f_1(z)\; e^{\gamma \widetilde{u}_1 + (1-\gamma)\widetilde{u}_2}\right\|_{\mathcal{C}^{0,\beta}_{\delta-2}(B_{r_\varepsilon}(z_3))} \le C r_\varepsilon^2$$

$$\left\|\Delta \widetilde{u}_2 + \rho^2 |z-p_3|^{2\alpha_3} |z-p_2|^{2\alpha_2} f_2(z)\; e^{\xi \widetilde{u}_2 + (1-\xi)\widetilde{u}_1}\right\|_{\mathcal{C}^{0,\beta}_{\mu-2}(B_{r_\varepsilon}(z_3))} \le C r_\varepsilon^2.$$

- In $\mathbf{B(z_2, r_\varepsilon)}$, we have

$$\begin{cases} \Delta\widetilde{u}_1 + \rho^2|z-p_1|^{2\alpha_1}|z-p_2|^{2\alpha_2} f_1(z) e^{\gamma\widetilde{u}_1+(1-\gamma)\widetilde{u}_2} = \dfrac{8\ \varepsilon^2\tau_2^2}{\left(\varepsilon^2+|\tau_2(z-z_2)|^2\right)^2} \\ \qquad \times\left[\dfrac{|z-p_1|^{2\alpha_1}|z-p_2|^{2\alpha_2} f_1(z)}{|z_2-p_1|^{2\alpha_1}|z_2-p_2|^{2\alpha_2} f_1(z_2)} e^{\frac{(1-\gamma)(\gamma+\xi-1)}{2-\gamma-\xi}\left(\frac{1+\alpha_1}{\gamma} G(z,p_1)-\frac{1}{\xi} G(z,z_3)\right)} - 1\right] \\ \Delta\widetilde{u}_2 + \rho^2|z-p_3|^{2\alpha_3}|z-p_2|^{2\alpha_2} f_2(z) e^{\xi\widetilde{u}_2+(1-\xi)\widetilde{u}_1} = \dfrac{8\ \varepsilon^2\tau_2^2}{\left(\varepsilon^2+|\tau_2(z-z_2)|^2\right)^2} \\ \qquad \times\left[\dfrac{|z-p_3|^{2\alpha_3}|z-p_2|^{2\alpha_2} f_2(z)}{|z_2-p_3|^{2\alpha_3}|z_2-p_2|^{2\alpha_2} f_2(z_2)} e^{\frac{(1-\xi)(\gamma+\xi-1)}{2-\gamma-\xi}\left(-\frac{1+\alpha_1}{\gamma} G(z,p_1)+\frac{1}{\xi} G(z,z_3)\right)} - 1\right]. \end{cases}$$

Then, for $r = |z - z_2|$, we fix $\mu \in (1, 2)$ and using the condition (25) given in Theorem 3, we have

$$\left\|\Delta\widetilde{u}_1 + \rho^2|z-p_1|^{2\alpha_1}|z-p_2|^{2\alpha_2} f_1(z)\ e^{\gamma\widetilde{u}_1+(1-\gamma)\widetilde{u}_2}\right\|_{\mathcal{C}^{0,\beta}_{\mu-2}(B_{r_\varepsilon}(z_2))} \le C r_\varepsilon^2$$

$$\left\|\Delta\widetilde{u}_2 + \rho^2|z-p_3|^{2\alpha_3}|z-p_2|^{2\alpha_2} f_2(z)\ e^{\xi\widetilde{u}_2+(1-\xi)\widetilde{u}_1}\right\|_{\mathcal{C}^{0,\beta}_{\mu-2}(B_{r_\varepsilon}(z_2))} \le C r_\varepsilon^2.$$

• **In $\mathbf{B(z_2, r_\varepsilon)}$**: Using the following transformation

$$\begin{cases} v_1(z) &= u_1(\frac{\varepsilon}{\tau_2}z) + \frac{4}{\gamma} \log\varepsilon - \frac{2}{\gamma} \log(\frac{\tau_2(1+\varepsilon^2)}{2}) \\ v_2(z) &= u_2(\frac{\varepsilon}{\tau_2}z) + \frac{4}{\xi} \log\varepsilon - \frac{2}{\xi} \log(\frac{\tau_2(1+\varepsilon^2)}{2}). \end{cases} \quad (68)$$

Then the system (38) can be written as

$$\begin{cases} -\Delta v_1 &= 2|\frac{\varepsilon}{\tau_2}z - p_1|^{2\alpha_1}|\frac{\varepsilon}{\tau_2}z - p_2|^{2\alpha_2} f_1(\frac{\varepsilon}{\tau_2}z)e^{\gamma v_1 + (1-\gamma)v_2} \quad \text{in } B_{R_\varepsilon^2}(z_2) \\ -\Delta v_2 &= 2|\frac{\varepsilon}{\tau_2}z - p_3|^{2\alpha_3}|\frac{\varepsilon}{\tau_2}z - p_2|^{2\alpha_2} f_2(\frac{\varepsilon}{\tau_2}z)e^{\xi v_2 + (1-\xi)v_1} \quad \text{in } B_{R_\varepsilon^2}(z_2), \end{cases} \quad (69)$$

where $R_\varepsilon^2 := \dfrac{\tau_2 r_\varepsilon}{\varepsilon}$.

We look for a solution of (69) of the form

$$
\begin{cases}
v_1(z) = u_{\varepsilon,\tau_2}(z - z_2) + \frac{(1+\alpha_1)(1-\gamma)}{\gamma(2-\gamma-\xi)} G(\frac{\varepsilon}{\tau_2} z, p_1) - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G(\frac{\varepsilon}{\tau_2} z, z_3) \\
\qquad - \frac{\xi}{\gamma+\xi-1} \log\left(|\frac{\varepsilon}{\tau_2} z_2 - p_1|^{2\alpha_1} |\frac{\varepsilon}{\tau_2} z_2 - p_2|^{2\alpha_2} f_1(\frac{\varepsilon}{\tau_2} z_2)\right) \\
\qquad + \frac{1-\gamma}{\gamma+\xi-1} \log\left(|\frac{\varepsilon}{\tau_2} z_2 - p_3|^{2\alpha_3} |\frac{\varepsilon}{\tau_2} z_2 - p_2|^{2\alpha_2} f_2(\frac{\varepsilon}{\tau_2} z_2)\right) \\
\qquad + H^{int}(\varphi_1^2, (z - z_2)/R_\varepsilon^2) + h_1^2(z) \\
v_2(z) = u_{\varepsilon,\tau_2}(z - z_2) - \frac{(1+\alpha_1)(1-\xi)}{\gamma(2-\gamma-\xi)} G(\frac{\varepsilon}{\tau_2} z, p_1) + \frac{1-\xi}{\xi(2-\gamma-\xi)} G(\frac{\varepsilon}{\tau_2} z, z_3) \\
\qquad + \frac{1-\xi}{\gamma+\xi-1} \log\left(|\frac{\varepsilon}{\tau_2} z_2 - p_1|^{2\alpha_1} |\frac{\varepsilon}{\tau_2} z_2 - p_2|^{2\alpha_2} f_1(\frac{\varepsilon}{\tau_2} z_2)\right) \\
\qquad - \frac{\gamma}{\gamma+\xi-1} \log\left(|\frac{\varepsilon}{\tau_2} z_2 - p_3|^{2\alpha_3} |\frac{\varepsilon}{\tau_2} z_2 - p_2|^{2\alpha_2} f_2(\frac{\varepsilon}{\tau_2} z_2)\right) \\
\qquad + H^{int}(\varphi_2^2, (z - z_2)/R_\varepsilon^2) + h_2^2(z).
\end{cases}
\tag{70}
$$

This amounts to solve the equations

$$
\begin{cases}
\mathbb{L}h_1^2 = \frac{8}{(1+r^2)^2}\left[\frac{|\frac{\varepsilon}{\tau_2}z-p_1|^{2\alpha_1}|\frac{\varepsilon}{\tau_2}z-p_2|^{2\alpha_2}f_1(\frac{\varepsilon}{\tau_2}z)}{|\frac{\varepsilon}{\tau_2}z_2-p_1|^{2\alpha_1}|\frac{\varepsilon}{\tau_2}z_2-p_2|^{2\alpha_2}f_1(\frac{\varepsilon}{\tau_2}z_2)}e^{\frac{(1-\gamma)(\gamma+\xi-1)}{2-\gamma-\xi}\left(\frac{1+\alpha_1}{\gamma}G(\frac{\varepsilon}{\tau_2}z,p_1)-\frac{1}{\xi}G(\frac{\varepsilon}{\tau_2}z,z_3)\right)}\right. \\
\qquad \left.\times e^{\gamma\left(h_1^2+H_1^{int,2}\right)+(1-\gamma)\left(h_2^2+H_2^{int,2}\right)}-h_1^2-1\right] \\
\mathbb{L}h_2^2 = \frac{8}{(1+r^2)^2}\left[\frac{|\frac{\varepsilon}{\tau_2}z-p_3|^{2\alpha_3}|\frac{\varepsilon}{\tau_2}z-2_3|^{2\alpha_2}f_2(\frac{\varepsilon}{\tau_2}z)}{|\frac{\varepsilon}{\tau_2}z_2-p_3|^{2\alpha_3}|\frac{\varepsilon}{\tau_2}z_2-p_2|^{2\alpha_2}f_2(\frac{\varepsilon}{\tau_2}z_2)}e^{\frac{(1-\xi)(\gamma+\xi-1)}{2-\gamma-\xi}\left(-\frac{1+\alpha_1}{\gamma}G(\frac{\varepsilon}{\tau_2}z,p_1)+\frac{1}{\xi}G(\frac{\varepsilon}{\tau_2}z,z_3)\right)}\right. \\
\qquad \left.\times e^{\xi\left(h_2^2+H_2^{int,2}\right)+(1-\xi)\left(h_1^2+H_1^{int,2}\right)}-h_2^2-1\right].
\end{cases}
\tag{71}
$$

We obtain the following proposition.

Proposition

Given $\kappa > 0$, $\alpha_2 \notin \mathbb{N}$, γ_0, ξ_0 and μ in $(0,1)$. Assume that the conditions (1) and (?) are satisfied, then there exist $\varepsilon_\kappa > 0$ and $c_\kappa > 0$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$,$\gamma \in (\gamma_0, 1)$ and $\xi \in (\xi_0, 1)$, there exists a unique (h_1^2, h_2^2) solution of (71) such that

$$\|(h_1^2, h_2^2)\|_{\mathcal{C}_\mu^{2,\beta}(\mathbb{R}^2)\times\mathcal{C}_\mu^{2,\beta}(\mathbb{R}^2)} \leqslant c_\kappa r_\varepsilon^2.$$

Hence (v_1, v_2) defined by (70) solves (69) in $B_{R_\varepsilon^2}(z_2)$.

The nonlinear exterior problem

We summarize this in the following proposition

Proposition

Given $\kappa > 0$, $\delta \in (0,1)$, $\mu \in (1/(2+\bar{\alpha}), \delta)$ and assume that the condition (24) is satisfied, there exists $\varepsilon_\kappa > 0$ (depending on κ) such that for any $\varepsilon \in (0, \varepsilon_\kappa)$, λ_i satisfying (64), any functions $\tilde{\varphi}_j^i$ satisfying (37) and (64), there exists a unique $(\tilde{v}_1, \tilde{v}_2)$ solution of (62) so that for $\mathbf{v_k}$ $(k = 1, 2)$ defined by

$$\begin{cases} \mathbf{v_1} & := \dfrac{1+\lambda_1+\alpha_1}{\gamma} G(\cdot, p_1) + (1+\lambda_2) G(\cdot, \tilde{z}_2) + \chi_{r_0}(\cdot - p_1) H_1^{ext}(\tilde{\varphi}_1^1; (\cdot - p_1)/r_{\varepsilon,\alpha}) \\ & \quad + \chi_{r_0}(\cdot - \tilde{z}_2) H_1^{ext}(\tilde{\varphi}_1^2; (\cdot - \tilde{z}_2)/r_\varepsilon) + \chi_{r_0}(\cdot - \tilde{z}_3) H_1^{ext}(\tilde{\varphi}_1^3; (\cdot - \tilde{z}_3)/r_\varepsilon) + \tilde{v}_1 \\ \mathbf{v_2} & := \dfrac{1+\lambda_3}{\xi} G(\cdot, \tilde{z}_3) + (1+\lambda_2) G(\cdot, \tilde{z}_2) + \chi_{r_0}(\cdot - p_1) H_2^{ext}(\tilde{\varphi}_2^1; (\cdot - p_1)/r_{\varepsilon,\alpha}) \\ & \quad + \chi_{r_0}(\cdot - \tilde{z}_2) H_2^{ext}(\tilde{\varphi}_2^2; (\cdot - \tilde{z}_2)/r_\varepsilon) + \chi_{r_0}(\cdot - \tilde{z}_3) H_2^{ext}(\tilde{\varphi}_2^3; (\cdot - \tilde{z}_3)/r_\varepsilon) + \tilde{v}_2, \end{cases}$$

solve (60) in $\bar{\Omega} - \left\{ B_{r_{\varepsilon,\alpha}}(p_1) \cup B_{r_\varepsilon}(\tilde{z}_2) \cup B_{r_\varepsilon}(\tilde{z}_3) \right\}$. In addition, we have
$\|\tilde{v}_k\|_{C_\nu^{2,\beta}(\bar{\Omega}^*(\tilde{\mathbf{p}}))} \leqslant 2\bar{c}_\kappa r_{\varepsilon,\alpha}^2$.

Near $\partial B_{r_\varepsilon}(\tilde z_2)$, it amounts to find the boundary data and the parameters so that for $i=1,2$

$$\Big((u_{int,i}-u_{ext,i})\Big)(\tilde z_2+r_\varepsilon\cdot)=0$$
$$\partial_r\Big((u_{int,i}-u_{ext,i})\Big)(\tilde z_2+r_\varepsilon\cdot)=0. \tag{72}$$

Then the systems found by projecting (72) gather in these equalities

$$T^2_{i,\varepsilon}=\Big(t_2,\lambda_2,\varphi^2_{i,0},\tilde\varphi^2_{i,0},\varphi^2_{i,1},\tilde\varphi^2_{i,1},\bar\nabla\mathcal{E}_2(\tilde z_2,\tilde{\mathbf{p}}),\varphi^2_{i,\perp},\tilde\varphi^2_{i,\perp}\Big)=\mathcal{O}(r^2_{\varepsilon,\alpha}),\quad \text{for } i=1,2,$$

where

$$\begin{aligned}\mathcal{E}_2(z,\tilde{\mathbf{p}}):=\quad & H(z,\tilde z_2)+\frac{(1+\alpha_1)(1-\xi)}{\gamma(2-\gamma-\xi)}G(z,p_1)+\frac{1-\gamma}{\xi(2-\gamma-\xi)}G(z,\tilde z_3)\\ &+\frac{\xi}{\gamma+\xi-1}\log\left(|\tilde z_2-p_1|^{2\alpha_1}|\tilde z_2-p_2|^{2\alpha_2}f_1(\tilde z_2)\right)\\ &-\tfrac{1-\gamma}{\gamma+\xi-1}\log\left(|\tilde z_2-p_3|^{2\alpha_3}|\tilde z_2-p_2|^{2\alpha_2}f_2(\tilde z_2)\right)\end{aligned}$$

and

$$t_2=\frac{1}{\log r_\varepsilon}\Big[2\log\tau_2+\mathcal{E}_2(\tilde z_2,\tilde{\mathbf{p}})\Big].$$

THANK YOU FOR YOUR ATTENTION