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Tracking Control of Chained Systems

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Outline

Introduction

Fig 1: Mobile Robot

Fig 2: Vacuum

Problem

• Construct a control algorithm which assure the tracking of a feasible trajectory of the chained form system.

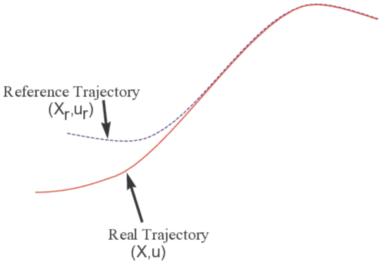

Figure: Tracking trajectory.

Let $|.|$ denotes the Euclidean norm and let *sgn* designates the sign function ($sgn(x) = \frac{x}{|x|}$ if $x \neq 0$ and $sgn(0) = 0$).

Definition [1]

The control system

$$\dot{x} = Y(x, u); \quad Y(0,0) = 0, \tag{1}$$

where $x \in \mathbb{R}^n$ *is the state,* $u \in \mathbb{R}^m$ *the control and* $Y \in C^0(\mathbb{R}^{m+n}, \mathbb{R}^n)$, *is said locally(resp. globally) polynomially stabilizable by means of continuous time-varying feedback laws if there exists* $u \in C^0(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^m)$ *satisfying*

$$u(t,0) = 0; \quad \forall t \geq 0, \tag{2}$$

such that $0 \in \mathbb{R}^n$ *is locally (resp. globally) polynomially stable for the closed-loop system* $\dot{x} = Y(x, u(t,x))$.

Theorem 1 [1]

Consider the nonlinear continuous time-varying dynamical system

$$\dot{x} = X(x,t); \quad X(0,t) = 0, \forall t \geq 0 \tag{3}$$

$x \in \mathbb{R}^n$ *is the state where* $X : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ *is a continuous vector field. Assume that there exists a* $C^1 - function\, V$*:[0,+∞[*$\times \mathbb{R}^n \to \mathbb{R}$*, some positive constants* c_1,c_2,r_1 *and* r_2 *such that,*

(1) *for every* $x \in \mathbb{R}^n$*, the Lyapunov function* V *satisfies*

$$c_1 \left|x\right|^{r_1} \leq V(t,x) \leq c_2 \left|x\right|^{r_2} \tag{4}$$

(2) *there exists a continuous and nonnegative function* $f : [0, +\infty[\to (0, +\infty)$ *with the growth condition:* $\int_0^t f(s)ds \geq kt^r$ *where* r *and* k *are two nonnegative constants such that*

$$\dot{V}(t,x) \leq -cf(t)V^{\alpha}(t,x), \alpha > 1 \text{ and } c > 0, \tag{5}$$

then $0 \in \mathbb{R}^n$ *is globally polynomially stable.*

Theorem 2 [3]

If the control system

$$\dot{x} = f(x, u),$$

is polynomially stabilizable with a feedback law in C^1.
Then the dynamic extension

$$\dot{x} = f(x, y)$$
$$\dot{y} = v,$$

is polynomially stabilizable with a feedback law in C^0

Control Design of Chained Systems

Nonlinear chained system

The control chained system is written as follows:

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_1 x_3 \\ \dot{x}_3 &= u_1 x_4 \\ &\vdots \\ \dot{x}_{n-1} &= u_1 x_n \\ \dot{x}_n &= u_2. \end{aligned} \tag{6}$$

The state vector is given by $(x_1, x_2, \cdots, x_{n-1}, x_n) \in \mathbb{R}^n$.
The vector $(u_1, u_2) \in \mathbb{R}^2$ represents the control.

The basic aim is to present an explicit design of control laws u_1 and u_2 such that the chained form system tracks the time reference position.

Reference system

• $(x_{1r}(t), x_{2r}(t), \cdots, x_{(n-1)r}(t), x_{nr}(t))$ describes the feasible reference trajectory of a chained form system.
• The reference control vector is $(u_{1r}(t), u_{2r}(t))$.

$$
\begin{aligned}
\dot{x}_{1r} &= u_{1r} \\
\dot{x}_{2r} &= u_{1r} x_{3r} \\
\dot{x}_{3r} &= u_{1r} x_{4r} \\
&\vdots \\
\dot{x}_{(n-1)r} &= u_{1r} x_{nr} \\
\dot{x}_{nr} &= u_{2r}.
\end{aligned}
\tag{7}
$$

Note that $u_{1r}(t) \neq 0$ and $u_{2r}(t) \neq 0$; $\forall t \geq 0$.

Error system

Let's note that: $e_i = x_i - x_{ir}$ with $1 \leq i \leq n$ and $v_j = u_j - u_{jr}$ with $1 \leq j \leq 2$.

Therefore, differentiating $(e_1, e_2, \cdots, e_n)$ along the trajectories (6) and (7) with respect to time, we obtain the following error model:

$$\begin{aligned} \dot{e}_1 &= v_1 \\ \dot{e}_2 &= v_1(e_3 + x_{3r}) + u_{1r} e_3 \\ \dot{e}_3 &= v_1(e_4 + x_{4r}) + u_{1r} e_4 \\ &\vdots \\ \dot{e}_{(n-1)} &= v_1(e_n + x_{nr}) + u_{1r} e_n \\ \dot{e}_n &= v_2. \end{aligned} \tag{8}$$

The main idea is to treat the error system as two connected subsystems $\sum_1$ and $\sum_2$.

$$\Sigma_1 : \dot{e}_1 = v_1 \quad ;$$

$$\Sigma_2 : \begin{cases} \dot{e}_2 & = v_1(e_3 + x_{3r}) + u_{1r}e_3 \\ \dot{e}_3 & = v_1(e_4 + x_{4r}) + u_{1r}e_4 \\ \vdots & \\ \dot{e}_{(n-1)} & = v_1(e_n + x_{nr}) + u_{1r}e_n \\ \dot{e}_n & = v_2. \end{cases}$$

To remove this coupling, we stabilize the subsystem $\sum_1$ in fixed-time faster than the remaining error variables $(e_2, \cdots, e_n)$. Then we focus on stabilizing the subsystem $\sum_2$

Stabilization of $\sum_1$

Lemma

Under the static feedback law $v_1(e_1) = -sgn(e_1)|e_1|^{1/2} - sgn(e_1)|e_1|^{3/2}$, *the one-dimensional control system* $\sum_1$ *is fixed- time stable.*
More precisely $e_1(t) = 0; \forall t \geq \pi$.

Proof:

In closed-loop, we have:

$$\dot{e}_1 = -sgn(e_1)|e_1|^{1/2} - sgn(e_1)|e_1|^{3/2}. \tag{9}$$

We fixed $e_1^0 = e_1(t_0), \;\; t_0 > 0,$

then the solution of (9) *is given by:*

$$e_1(t) = \begin{cases} sgn(e_1^0)tg^2(arctg(\sqrt{|e_1^0|}) - t/2); & for \;\; t \leq 2arctg(\sqrt{(|e_1^0|})) \\ 0; & else, \end{cases}$$

then $t \leq 2arctg(\sqrt{(|e_1^0|})) \leq \pi.$

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Then for $t > T_0 = \pi$ the dynamic of the $\sum_2$ is reduced to:

$$\begin{cases} \dot{e}_2 & = u_{1r} e_3 \\ \dot{e}_3 & = u_{1r} e_4 \\ \vdots & \\ \dot{e}_{(n-1)} & = u_{1r} e_n \\ \dot{e}_n & = v_2 \end{cases}$$

decoupling is thus carried out.

Application of backstepping approach $(n-3)$ times:

$$\begin{cases} \dot{e}_2 &= u_{1r} e_3 \\ \dot{e}_3 &= v_2^{(n-3,n-3)} \end{cases} \tag{10}$$

$$\begin{aligned} v_2 &= -K_1(e_n - v_2^{(1)}) \\ v_2^{(1,1)} &= -K_2(e_{n-1} - v_2^{(2)}) \\ &\vdots \\ v_2^{(n-5,n-5)} &= -K_{n-4}(e_5 - v_2^{(n-4)}) \\ v_2^{(n-4,n-4)} &= -K_{n-3}(e_4 - v_2^{(n-3)}), \end{aligned} \tag{11}$$

where:
$v_2^{(n-s,n-s)} = u_{1r} v_2^{(n-s)}$ and K_{n-s} are positive and large enough values;
$\forall \ 3 \leq s \leq n-1$

Proposition

Let $p > 1$ and $k > \frac{1}{2}$ are two nonnegative odd rational numbers. Assuming that u_{1r} is continuously differentiable and satisfies the constraint $u_{1r}(t) > 0$; $\forall t \geq \pi$.
Under the choice of the following feedback law

$$v_2^{(n-3,n-3)} = -u_{1r}e_2^{2k-1} - f^{'}(t)e_2^p - pu_{1r}f(t)e_2^{p-1}e_3 - u_{1r}f(t)(e_3 + f(t)e_2^p)^{1+\frac{p-1}{k}}, \quad (12)$$

with $f : [\pi, +\infty[\longrightarrow]0, +\infty[$ is a C^{n-2}-function with the constraint $\int_\pi^t f(s)ds \geq \mu t^r, \mu, r > 0$.
The reduced system (10) is globally polynomially stabilizable for $t \geq \pi$.

Proof:

The construction of the polynomial feedback law that guarantees the stabilization of (10) can be obtained through the following candidate positive definite Lyapunov function

$$V_{k,p}(e_3, e_2, t) = \tfrac{1}{2k} e_2^{2k} + \tfrac{1}{2}(e_3 + f(t)e_2^p)^2 \geq 0$$

with $k > \frac{1}{2}$ *and* $p > 1$.

Differentiating V *along the solutions of the closed-loop system (10) we get:*

$$\begin{aligned}\dot{V}_{k,p}(e_3, e_2, t) = {} & u_{1r} e_2^{2k-1} e_3 + (e_3 + f(t)e_2^p)(v_2^{(n-3,n-3)} \\ & + f^{'}(t)e_2^p + p u_{1r} f(t) e_2^{p-1} e_3). \end{aligned} \qquad (13)$$

If we replace $v_2^{(n-3,n-3)}$ *by the following expression:*

$$v_2^{(n-3,n-3)} = -f^{'}(t)e_2^p - p u_{1r} f(t) e_2^{p-1} e_3 - u_{1r} e_2^{2k-1} + v_2^{'}, \qquad (14)$$

we obtain:

$$\dot{V}_{k,p}(e_3, e_2, t) = u_{1r} e_2^{2k-1} e_3 + (e_3 + f(t) e_2^p)(-u_{1r} e_2^{2k-1} + v_2^{'}). \tag{15}$$

With the following choice of $v_2^{'}$*:*

$$v_2^{'} = -u_{1r} f(t)(e_3 + f(t) e_2^p)^{1+\frac{p-1}{k}}, \tag{16}$$

we will have:

$$\dot{V}_{k,p}(e_3, e_2, t) = -f(t) u_{1r} [(e_3 + f(t) e_2^p)^{2+\frac{p-1}{k}} + e_2^{2k+p-1}], \tag{17}$$

implies that:

$$\begin{cases} \left(\frac{-1}{f(t)u_{1r}}\dot{V}_{k,p}\right)^{\frac{2k}{2k+p-1}} \geq \frac{1}{2}(e_3 + f(t)e_2^p)^2 \\ \left(\frac{-1}{f(t)u_{1r}}\dot{V}_{k,p}\right)^{\frac{2k}{2k+p-1}} \geq \frac{1}{2k}e_2^{2k}. \end{cases} \tag{18}$$

If we sum the two preceding inequalities, we get:

$$\dot{V}_{k,p} \leq -u_{1r}\left(\frac{1}{2}\right)^{\alpha} f(t) V_{k,p}^{\alpha}, \tag{19}$$

with $\alpha = \frac{2k+p-1}{2k} > 1$.

We choose the reference control $u_{1r} = \frac{g(t)}{f(t)}$ *where* g *is a non-negative* C^0*-function satisfying the constraint* $\int_{\pi}^{t} g(s)ds \geq \delta t^m, \delta, m > 0$.

$$\dot{V}_{k,p} \leq -cg(t)V_{k,p}^{\alpha} \tag{20}$$

Hence, the subsystem (10) is globally polynomially stabilizable.

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Application: Unicycle Mobile Robots

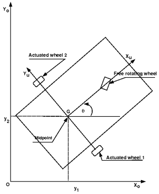

Figure: Unicycle Mobile Robots.

The study model motion is governed by the following well-known non-linear driftless control system:

$$\begin{aligned} \dot{y}_1 &= d_1 cos(\theta) \\ \dot{y}_2 &= d_1 sin(\theta) \\ \dot{\theta} &= d_2, \end{aligned} \tag{21}$$

where:
• y_1 and y_2 are the instantaneous position planar coordinates of G.
• θ is the angle between the heading direction and the x-axis.
• (d_1, d_2) the pair of input variables which represents respectively the forward command and the steering control.

Let us consider the following change of variables:

$$\begin{aligned} x_1 &= \theta \\ x_2 &= y_1 sin(\theta) - y_2 cos(\theta) \\ x_3 &= y_1 cos(\theta) + y_2 sin(\theta). \end{aligned}$$

Simple calculation yields

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_1 x_3 \\ \dot{x}_3 &= u_2, \end{aligned} \tag{22}$$

with $u_1 = d_2$ and $u_2 = d_1 - d_2 x_2$.

The error system is given by:

$$\begin{aligned} \dot{e}_1 &= v_1 \\ \dot{e}_2 &= v_1(e_3 + x_{3r}) + u_{1r}e_3 \\ \dot{e}_3 &= v_2, \end{aligned} \tag{23}$$

where $e_i = x_i - x_{ir}$ for $1 \leq i \leq 3$, $v_j = u_j - u_{jr}$ for $1 \leq j \leq 2$.

- **1^{st}step** : Stabilization of the first variable of (23) in fixed-time:

$$v_1(e_1) = -sgn(e_1)|e_1|^{1/2} - sgn(e_1)|e_1|^{3/2} \;\; when \;\; t < \pi \tag{24}$$

- **2^{nd}step** :Stabilization of the remaining variables:

Let $p > 1$ and $k > \frac{1}{2}$ are two nonnegative odd rational numbers.
Let $u_{1r} > 0$, $\forall t > 0$ and is continuously differentiable, under the continuous time varying feedback law

$$v_2 = -u_{1r} e_2^{2k-1} - f^{'}(t) e_2^p - p u_{1r} f(t) e_2^{p-1} e_3 - u_{1r} f(t)(e_3 + f(t) e_2^p)^{1+\frac{p-1}{k}}. \quad (25)$$

With $f : [\pi, +\infty[\longrightarrow]0, +\infty[$ is a C^1-function with the constraint $\int_\pi^t f(s) ds \geq \mu t^r, \mu, r > 0$.
the subsystem is globally polynomially stabilizable for $t \geq \pi$.

Simulation results

- The reference laws $d_{1r} = 1$ and $d_{2r} = cos(t) + 2$.
- The reference initial conditions: $y_{1r}(0) = y_{2r}(0) = 0$ and $\theta_r(0) = 0$.
- The parameters: k and p are such that $(k, p) = (1, 3)$.
- $f(t) = 1 + t^5; \forall t \geq \pi$

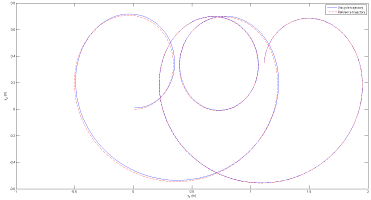

Fig 1: Tracking trajectory $y_1 = 0.01$ and $y_2 = 0.01$

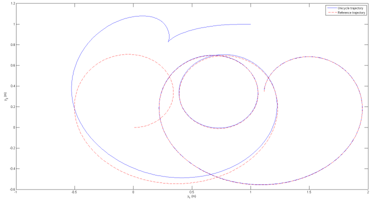

Fig 2: Tracking trajectory $y_1 = 1$ and $y_2 = 1$

Conclusion and perspective

• Tracking time-varying feedback law combined with fixed-time control design problem is investigated for a chained system.

• Use this approach for a chained system including measurment noise.

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THANK YOU FOR YOUR ATTENTION