Tracking Control of Chained Systems

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Introduction
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Fig 1: Mobile Robot

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Problem

- Construct a control algorithm which assure the tracking of a feasible trajectory of the chained form system.

**Reference Trajectory**
$(X_r, u_r)$

**Real Trajectory**
$(X, u)$

**Figure:** Tracking trajectory.
Let \(|.|\) denotes the Euclidean norm and let \(\text{sgn}\) designates the sign function 
\(\text{sgn}(x) = \frac{x}{|x|}\) if \(x \neq 0\) and \(\text{sgn}(0) = 0\).

**Definition [1]**

The control system

\[
\dot{x} = Y(x, u); \quad Y(0, 0) = 0,
\]

where \(x \in \mathbb{R}^n\) is the state, \(u \in \mathbb{R}^m\) the control and \(Y \in C^0(\mathbb{R}^{m+n}, \mathbb{R}^n)\), is said locally (resp. globally) polynomially stabilizable by means of continuous time-varying feedback laws if there exists \(u \in C^0(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^m)\) satisfying

\[
u(t, 0) = 0; \quad \forall t \geq 0,
\]

such that \(0 \in \mathbb{R}^n\) is locally (resp. globally) polynomially stable for the closed-loop system \(\dot{x} = Y(x, u(t, x))\).
Theorem 1 [1]

Consider the nonlinear continuous time-varying dynamical system

\[ \dot{x} = X(x, t); \quad X(0, t) = 0, \forall t \geq 0 \]  

(3)

\( x \in \mathbb{R}^n \) is the state where \( X : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n \) is a continuous vector field.

Assume that there exists a \( C^1 \) function \( V : [0, +\infty[ \times \mathbb{R}^n \rightarrow \mathbb{R} \), some positive constants \( c_1, c_2, r_1 \) and \( r_2 \) such that,

1. for every \( x \in \mathbb{R}^n \), the Lyapunov function \( V \) satisfies

\[ c_1 \|x\|^{r_1} \leq V(t, x) \leq c_2 \|x\|^{r_2} \]  

(4)

2. there exists a continuous and nonnegative function \( f : [0, +\infty[ \rightarrow (0, +\infty) \) with the growth condition: \( \int_0^t f(s) ds \geq kt^r \) where \( r \) and \( k \) are two nonnegative constants such that

\[ \dot{V}(t, x) \leq -cf(t)V^\alpha(t, x), \alpha > 1 \text{ and } c > 0, \]  

(5)

then \( 0 \in \mathbb{R}^n \) is globally polynomially stable.
Theorem 2 [3]

If the control system

\[ \dot{x} = f(x, u), \]

is polynomially stabilizable with a feedback law in \( C^1 \).

Then the dynamic extension

\[ \begin{align*}
\dot{x} &= f(x, y) \\
\dot{y} &= v,
\end{align*} \]

is polynomially stabilizable with a feedback law in \( C^0 \).
Control Design of Chained Systems
The control chained system is written as follows:

\[
\begin{align*}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_1 x_3 \\
\dot{x}_3 &= u_1 x_4 \\
&\quad \vdots \\
\dot{x}_{n-1} &= u_1 x_n \\
\dot{x}_n &= u_2.
\end{align*}
\]

The state vector is given by \((x_1, x_2, \cdots, x_{n-1}, x_n) \in \mathbb{R}^n\).

The vector \((u_1, u_2) \in \mathbb{R}^2\) represents the control.

The basic aim is to present an explicit design of control laws \(u_1\) and \(u_2\) such that the chained form system tracks the time reference position.
Reference system

- \((x_{1r}(t), x_{2r}(t), \cdots, x_{(n-1)r}(t), x_{nr}(t))\) describes the feasible reference trajectory of a chained form system.
- The reference control vector is \((u_{1r}(t), u_{2r}(t))\).

\[
\begin{align*}
\dot{x}_{1r} &= u_{1r} \\
\dot{x}_{2r} &= u_{1r} x_{3r} \\
\dot{x}_{3r} &= u_{1r} x_{4r} \\
&\vdots \\
\dot{x}_{(n-1)r} &= u_{1r} x_{nr} \\
\dot{x}_{nr} &= u_{2r}.
\end{align*}
\]  

(7)

Note that \(u_{1r}(t) \neq 0\) and \(u_{2r}(t) \neq 0; \forall t \geq 0\).
Let's note that: $e_i = x_i - x_{ir}$ with $1 \leq i \leq n$ and $v_j = u_j - u_{jr}$ with $1 \leq j \leq 2$.

Therefore, differentiating $(e_1, e_2, \cdots, e_n)$ along the trajectories (6) and (7) with respect to time, we obtain the following error model:

\[
\begin{align*}
\dot{e}_1 &= v_1 \\
\dot{e}_2 &= v_1(e_3 + x_{3r}) + u_{1r}e_3 \\
\dot{e}_3 &= v_1(e_4 + x_{4r}) + u_{1r}e_4 \\
\vdots \\
\dot{e}_{(n-1)} &= v_1(e_n + x_{nr}) + u_{1r}e_n \\
\dot{e}_n &= v_2. \\
\end{align*}
\] (8)
The main idea is to treat the error system as two connected subsystems $\Sigma_1$ and $\Sigma_2$.

$$\Sigma_1 : \dot{e}_1 = v_1 ;$$

$$\Sigma_2 : \begin{cases}
\dot{e}_2 &= v_1(e_3 + x_{3r}) + u_{1r} e_3 \\
\dot{e}_3 &= v_1(e_4 + x_{4r}) + u_{1r} e_4 \\
\vdots
\dot{e}_{n-1} &= v_1(e_n + x_{nr}) + u_{1r} e_n \\
\dot{e}_n &= v_2.
\end{cases}$$

To remove this coupling, we stabilize the subsystem $\Sigma_1$ in fixed-time faster than the remaining error variables $(e_2, \cdots, e_n)$. Then we focus on stabilizing the subsystem $\Sigma_2$. 
Stabilization of $\Sigma_1$

**Lemma**

Under the static feedback law $v_1(e_1) = -\text{sgn}(e_1)|e_1|^{1/2} - \text{sgn}(e_1)|e_1|^{3/2}$, the one-dimensional control system $\Sigma_1$ is fixed-time stable. More precisely $e_1(t) = 0; \forall t \geq \pi$. 
Proof:

In closed-loop, we have:

\[
\dot{e}_1 = -\text{sgn}(e_1)|e_1|^{1/2} - \text{sgn}(e_1)|e_1|^{3/2}.
\]

(9)

We fixed \( e_1^0 = e_1(t_0), \ t_0 > 0, \)

then the solution of (9) is given by:

\[
e_1(t) = \begin{cases} 
\text{sgn}(e_1^0)tg^2(\text{arctg}(\sqrt{|e_1^0|}) - t/2); & \text{for } t \leq 2\text{arctg}(\sqrt{|e_1^0|}) \\
0; & \text{else,}
\end{cases}
\]

then \( t \leq 2\text{arctg}(\sqrt{|e_1^0|}) \leq \pi. \)
Then for $t > T_0 = \pi$ the dynamic of the $\sum_2$ is reduced to:

\[
\begin{align*}
\dot{\epsilon}_2 &= u_1 r \epsilon_3 \\
\dot{\epsilon}_3 &= u_1 r \epsilon_4 \\
\vdots \\
\dot{\epsilon}_{n-1} &= u_1 r \epsilon_n \\
\dot{\epsilon}_n &= v_2
\end{align*}
\]

decoupling is thus carried out.
Stabilization of $\Sigma_2$

Application of backstepping approach $(n-3)$ times:

$$\begin{cases} 
\dot{e}_2 = u_1 e_3 \\
\dot{e}_3 = v_2^{(n-3,n-3)} 
\end{cases} \quad (10)$$

$$v_2 = -K_1(e_n - v_2^{(1)})$$
$$v_2^{(1,1)} = -K_2(e_{n-1} - v_2^{(2)})$$
$$\vdots$$
$$v_2^{(n-5,n-5)} = -K_{n-4}(e_5 - v_2^{(n-4)})$$
$$v_2^{(n-4,n-4)} = -K_{n-3}(e_4 - v_2^{(n-3)}), \quad (11)$$

where:
$$v_2^{(n-s,n-s)} = u_1 r v_2^{(n-s)}$$ and $K_{n-s}$ are positive and large enough values;
$$\forall \ 3 \leq s \leq n-1$$
Proposition

Let \( p > 1 \) and \( k > \frac{1}{2} \) are two nonnegative odd rational numbers. Assuming that \( u_{1r} \) is continuously differentiable and satisfies the constraint \( u_{1r}(t) > 0; \forall t \geq \pi \).

Under the choice of the following feedback law

\[
v_2^{(n-3,n-3)} = -u_{1r} e_2^{2k-1} - f'(t)e_2^p - pu_{1r} f(t)e_2^{p-1}e_3 - u_{1r} f(t)(e_3 + f(t)e_2^p)^{1+\frac{p-1}{k}},
\]  

with \( f : [\pi, +\infty[ \longrightarrow ]0, +\infty[ \) is a \( C^{n-2} \)-function with the constraint \( \int_{\pi}^t f(s) ds \geq \mu t^r, \mu, r > 0 \).

The reduced system (10) is globally polynomially stabilizable for \( t \geq \pi \).
Proof:

The construction of the polynomial feedback law that guarantees the stabilization of (10) can be obtained through the following candidate positive definite Lyapunov function

\[ V_{k,p}(e_3, e_2, t) = \frac{1}{2k} e_2^{2k} + \frac{1}{2} (e_3 + f(t)e_2^{p})^2 \geq 0 \]

with \( k > \frac{1}{2} \) and \( p > 1 \).

Differentiating \( V \) along the solutions of the closed-loop system (10) we get:

\[
\dot{V}_{k,p}(e_3, e_2, t) = u_1 r e_2^{2k-1} e_3 + (e_3 + f(t)e_2^{p})(v_2^{(n-3,n-3)})
\]
\[
+ f'(t)e_2^{p} + p u_1 r f(t)e_2^{p-1} e_3. \tag{13}
\]

If we replace \( v_2^{(n-3,n-3)} \) by the following expression:

\[
v_2^{(n-3,n-3)} = -f'(t)e_2^{p} - p u_1 r f(t)e_2^{p-1} e_3 - u_1 r e_2^{2k-1} + v_2', \tag{14}
\]

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we obtain:

\[ \dot{V}_{k,p}(e_3, e_2, t) = u_1 r e_2^{2k-1} e_3 + (e_3 + f(t) e_2^p)(-u_1 r e_2^{2k-1} + v_2'). \] (15)

With the following choice of \( v_2' \):

\[ v_2' = -u_1 r f(t)(e_3 + f(t) e_2^p)^{1+\frac{p-1}{k}}, \] (16)

we will have:

\[ \dot{V}_{k,p}(e_3, e_2, t) = -f(t) u_1 r [(e_3 + f(t) e_2^p)^{2+\frac{p-1}{k}} + e_2^{2k+p-1}], \] (17)
implies that:

\[
\begin{cases}
\left( \frac{-1}{f(t)u_{1r}} \dot{V}_{k,p} \right)^{\frac{2k}{2k+p-1}} \geq \frac{1}{2} (e_3 + f(t)e_2^p)^2 \\
\left( \frac{-1}{f(t)u_{1r}} \dot{V}_{k,p} \right)^{\frac{2k}{2k+p-1}} \geq \frac{1}{2k} e_2^{2k}.
\end{cases}
\] (18)

If we sum the two preceding inequalities, we get:

\[
\dot{V}_{k,p} \leq -u_{1r} \left( \frac{1}{2} \right)^\alpha f(t)V_k^\alpha,
\] (19)

with \( \alpha = \frac{2k+p-1}{2k} > 1. \)

We choose the reference control \( u_{1r} = \frac{g(t)}{f(t)} \) where \( g \) is a non-negative \( C^0 \)-function satisfying the constraint \( \int_\pi^t g(s)ds \geq \delta t^m, \delta, m > 0. \)

\[
\dot{V}_{k,p} \leq -cg(t)V_k^\alpha
\] (20)

Hence, the subsystem (10) is globally polynomially stabilizable.
Application: Unicycle Mobile Robots
The study model motion is governed by the following well-known non-linear driftless control system:

\[
\begin{align*}
\dot{y}_1 &= d_1 \cos(\theta) \\
\dot{y}_2 &= d_1 \sin(\theta) \\
\dot{\theta} &= d_2,
\end{align*}
\]
where:
• $y_1$ and $y_2$ are the instantaneous position planar coordinates of $G$.
• $\theta$ is the angle between the heading direction and the x-axis.
• $(d_1, d_2)$ the pair of input variables which represents respectively the forward command and the steering control.

Let us consider the following change of variables:

$$
\begin{align*}
  x_1 &= \theta \\
  x_2 &= y_1 \sin(\theta) - y_2 \cos(\theta) \\
  x_3 &= y_1 \cos(\theta) + y_2 \sin(\theta).
\end{align*}
$$

Simple calculation yields

$$
\begin{align*}
  \dot{x}_1 &= u_1 \\
  \dot{x}_2 &= u_1 x_3 \\
  \dot{x}_3 &= u_2,
\end{align*}
$$

with $u_1 = d_2$ and $u_2 = d_1 - d_2 x_2$. 
The error system is given by:

\begin{align*}
\dot{e}_1 &= v_1 \\
\dot{e}_2 &= v_1(e_3 + x_{3r}) + u_{1r}e_3 \\
\dot{e}_3 &= v_2,
\end{align*}

(23)

where \( e_i = x_i - x_{ir} \) for \( 1 \leq i \leq 3 \), \( v_j = u_j - u_{jr} \) for \( 1 \leq j \leq 2 \).

• **1st step**: Stabilization of the first variable of (23) in fixed-time:

\[ v_1(e_1) = -sgn(e_1)|e_1|^{1/2} - sgn(e_1)|e_1|^{3/2} \text{ when } t < \pi \]

(24)
• **2^{nd} step**: Stabilization of the remaining variables:

Let $p > 1$ and $k > \frac{1}{2}$ are two nonnegative odd rational numbers. Let $u_{1r} > 0$, $\forall t > 0$ and is continuously differentiable, under the continuous time varying feedback law

$$v_2 = -u_{1r} e_2^{2k-1} - f'(t)e_2^p - p u_{1r} f(t)e_2^{p-1}e_3$$

$$- u_{1r} f(t)(e_3 + f(t)e_2^{p})^{1+\frac{p-1}{k}}. \quad (25)$$

With $f : [\pi, +\infty[ \longrightarrow 0, +\infty[ \text{ is a } C^1\text{-function with the constraint}$

$$\int_{\pi}^{t} f(s) ds \geq \mu t^r, \mu, r > 0.$$

the subsystem is globally polynomially stabilizable for $t \geq \pi$. 
Simulation results

• The reference laws $d_{1r} = 1$ and $d_{2r} = \cos(t) + 2$.
• The reference initial conditions: $y_{1r}(0) = y_{2r}(0) = 0$ and $\theta_{r}(0) = 0$.
• The parameters: $k$ and $p$ are such that $(k, p) = (1, 3)$.
• $f(t) = 1 + t^5; \forall t \geq \pi$
Fig 1: Tracking trajectory $y_1 = 0.01$ and $y_2 = 0.01$

Fig 2: Tracking trajectory $y_1 = 1$ and $y_2 = 1$
• Tracking time-varying feedback law combined with fixed-time control design problem is investigated for a chained system.

• Use this approach for a chained system including measurement noise.


THANK YOU FOR YOUR ATTENTION